Examples of Forced Symmetry-Breaking to Homoclinic Cycles in Three-Dimensional Euclidean-Invariant Systems

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Abstract

We study perturbations of cubic planforms, proving there exists perturbations giving homoclinic cycles between persistent steady states. Our results do not depend on the representation of the symmetry group of the lattice and are thus quite general.

The problem is studied using group theory rather than direct methods. We use the abstract action of the symmetry group of the perturbation on the group orbit to determine the existence of zero- and one-dimensional flow-invariant subspaces. The residual symmetry of the perturbation constrains the flows on these subspaces and, in certain cases, homoclinic cycles are guaranteed to exist.

Cubic planforms are physically interesting due to their relevance to certain physical systems. Applications to reaction-diffusion systems, nonlinear optical systems and the polyacrylamide methylene blue oxygen reaction are discussed.

1 Introduction

Many authors have considered pattern formation in two- and three-dimensional Euclidean-invariant systems. Equivariant bifurcation theory Golubitsky et al. [1988]; Golubitsky & Stewart [2002] provides powerful techniques for investigating the mathematical aspects of pattern-forming instabilities. The abstract results have been applied to a number of applications to different physical systems. Many authors have considered steady spatially-periodic patterns, or *planforms*. There is a large body of work concerning square planforms [Demircan & Seehafer, 2001; Dionne et al., 1997; Dionne & Golubitsky, 1992; Golubitsky et al., 1984], hexagonal planforms [Bresloff et al., 2001; Buzano & Golubitsky, 1983; Dionne et al., 1997; Dionne & Golubitsky, 1992] and more recently three-dimensional planforms, in particular, cubic planforms [Callahan & Knobloch, 1997, 1999; De Wit et al., 1992; Dionne, 1993; Gomes, 1999]. These abstract mathematical results have applications to many diverse physical systems, including: block copolymer melts [Bates & Fredrickson, 1999], visual hallucinations [Bresloff et al., 2001; Ermentrout & Cowan, 1979], the Bénard problem [Buzano & Golubitsky, 1983; Demircan & Seehafer, 2001; Eckert et al., 1998; Golubitsky et al., 1984; Schatz et al., 1999], reaction-diffusion systems [Callahan & Knobloch, 1999; Castets et al., 1990;

De Wit et al., 1992; Gomes, 1999; Gunaratne et al., 1994; Ouyang & Swinney, 1991; Turing, 1952; Walgraef et al., 1982], nonlinear optical systems [Degtiarev & Vorontsov, 1996; Komarova et al., 1997; Staliunas, 1998; Staliunas & J.Sánchez-Morcillo, 2000; Staliunas et al., 1997; Vorontsov & Samson, 1998; Vorontsov & Karpov, 1997; Vorontsov & Firth, 1994], Faraday waves [Edwards & Fauve, 1994, 1993; Fauve et al., 1992], microemulsions [Góźdź & Hołst, 1996] and the polyacrylamide methylene blue oxygen (PA-MBO) reaction [Kurin-Csörgei et al., 1998; Münster et al., 1996; Orbán et al., 1999; Steinbock et al., 1999]. Some of these phenomena are not limited to purely two-dimensional patterns, indeed evidence exists for three-dimensional patterns in several systems, for example, reaction-diffusion systems [De Wit et al., 1992; Gomes, 1999], nonlinear optical systems [Vorontsov & Samson, 1998; Staliunas, 1998] and the polyacrylamide methylene blue oxygen reaction [Kurin-Csörgei et al., 1998; Münster et al., 1996; Orbán et al., 1999; Steinbock et al., 1999].

The standard mathematical approach to these systems is equivariant bifurcation theory [Golubitsky et al., 1988]. These methods use Liapunov–Schmidt (or centre manifold) reduction to derive a finite-dimensional system of ordinary differential equations (ODEs), which is sometimes referred to as a 'Landau equation'. We assume that this reduction has been performed and consider a system of ODEs that are equivariant under the action of a compact Lie group $\Gamma \subseteq \mathbf{O}(n)$ acting on $\mathbb{R}^n$. That is, the system

$$\dot{x} = f(x) \tag{1}$$

satisfies

$$f(\gamma x) = \gamma f(x)$$

for all $\gamma \in \Gamma$. This approach, which exploits the symmetry of the system, gives generic results concerning the possible two- and three-dimensional planforms. A complete study of the so-called translation-free axial planforms has been completed [Dionne & Golubitsky, 1992; Dionne, 1993]. A planform is *translation-free* if the only translation that acts trivially is the identity translation.

Whilst equivariant bifurcation theory can produce results that agree well with physical observations, the modelling process naturally introduces symmetries that are only approximately present in the real system. If the real system is only a 'small' perturbation of the idealized model then we would expect a 'degree' of agreement between the model and physical system. It is desirable to understand how the addition of small symmetry-breaking terms to Eq. (1) influence the predictions of the model. This process is called *forced* *symmetry-breaking* or sometimes *explicit* or *system symmetry-breaking*. In the context of this paper we study how three-dimensional cubic planforms are affected by the addition of small symmetry-breaking effects. Such affects may enter a system in a number of ways: perturbations of the boundary conditions which break the assumed translational-invariance, or a chemical gradient in a reaction-diffusion system. To be precise, suppose Eq. (1) has a steady-state x_0 with $\Sigma \subseteq \Gamma$ symmetry. Then Γ-equivariance implies that

$$X_0 = \Gamma x_0$$

is a group orbit of steady states. Under generic conditions this group orbit is normally hyperbolic Field [1980]. We make the additional assumption that X_0 is asymptotically stable, although this isn't necessary for the methods we describe, we assume it for physical relevance. Let Δ be a Lie subgroup of Γ. Let g be Δ-equivariant and ε small. Consider the perturbation of Eq. (1) given by

$$F(x, \varepsilon) = f(x) + \varepsilon g(x).$$

If ε is sufficiently small then a theorem of Lauterbach & Roberts [1992] guarantees the existence of a flow-invariant manifold X_ε for the perturbed system. Moreover, X_ε is Δ-equivariantly diffeomorphic to X_0.

Several authors have discussed the above formulation of forced symmetry-breaking and applied it to different systems. Lauterbach & Roberts [1992], Lauterbach et al. [1996] and Maier-Paape & Lauterbach [2000] apply these methods to reaction-diffusion systems with spherical symmetry. Guyard & Lauterbach [1997] consider relative periodic orbits and apply their results to systems with spherical symmetry. Hou & Golubitsky [1997] consider two-dimensional reaction-diffusion systems on the square lattice and prove there exists an open set of perturbations that give asymptotically stable heteroclinic cycles. The work of Hou & Golubitsky [1997] was generalized by Parker et al. [2006a,b] to perturbations on the square and hexagonal lattice and is not limited to the 'fundamental representation' of the lattice. This paper continues this work with several three-dimensional examples. Specifically we consider cubic bifurcation problems each with $\mathbb{O} \oplus \mathbf{Z}_2^c \dotplus \mathbf{T}^3$ symmetry. These bifurcation problems occurs in the study of spatially periodic solutions to three-dimensional Euclidean-invariant systems on the cubic lattices. This problem was considered by Callahan & Knobloch [1997] and the authors completely classify the branching and stability of the 'axial' solutions in the fundamental representation of the group $\mathbb{O} \oplus \mathbf{Z}_2^c \dotplus \mathbf{T}^3$. A solution is said to be *axial* if it is guarantee to exist by the equivariant branching lemma [Golubitsky et al., 1988; Cicogna, 1981; Vanderbauwhede, 1982]. Dionne [1993] considers all (translation-free) representations of $\mathbb{O} \oplus \mathbf{Z}_2^c \dot{+} \mathbf{T}^3$ deriving all (translation-free) axial planforms supported by the three-dimensional cubic lattices. A representation is translation-free if there are no nontrivial translations that act trivially, and a solution is translation free if its symmetry group has trivial intersection with the three-torus of translations. We consider perturbations of translation free axial solutions on the simple (SC), face-centred (FCC) and body-centred (BCC) cubic lattices with $\mathbb{O} \oplus \mathbf{Z}_2^c$ symmetry. Our results apply to any translation free axial solution on the cubic lattices with symmetry isomorphic (but not necessarily conjugate) to $\mathbb{O} \oplus \mathbf{Z}_2^c$. More precisely, on each cubic lattice we show there exist perturbations with $\Delta \subseteq \mathbb{O}$ symmetry such that the perturbed flow has persistent steady states and homoclinic cycles to these steady states. Moreover, these results apply to any translation free irreducible representation of the lattice (with certain restrictions). Since our results are derived in a generic framework using only group theory they are model-independent applying to any system of PDEs that support three-dimensional spatially-periodic time-independent solutions.

Organization of the paper. We formulate our abstract problem in Sec. 2. That is, given a Γ-equivariant bifurcation problem with a steady-state solution with $\Sigma \subseteq \Gamma$ symmetry, we examine how the group orbit X_0 behaves when terms with $\Delta \subseteq \Gamma$ symmetry are added to the bifurcation problem. This is studied via the abstract action of Δ on the homogeneous space Γ/Σ, which is Γ-equivariantly diffeomorphic to the orbit space. Certain subspaces of Γ/Σ are flow-invariant and we consider the collection $\mathscr{C}_\Delta$ that consists of those subspaces homeomorphic to a point or a circle. The residual Δ symmetry has an induced action on $\mathscr{C}_\Delta$ which restricts Δ-equivariant flows on the elements of $\mathscr{C}_\Delta$. The elements of $\mathscr{C}_\Delta$ collectively form the skeleton of the abstract problem. The skeleton describes all steady states and connecting orbits that are forced to exist by the residual Δ symmetry. The projected skeleton is the quotient of the skeleton by the Δ-action and a general argument in topology proves that if the projected skeleton possesses a connected set in a one-dimensional stratum then the perturbed flow has a homoclinic cycle. In Sec. 2.2 we recall the general process employed to reduce a Euclidean-invariant system of PDEs to a system of ODEs equivariant under the action of a compact Lie group.

Section 3 presents a perturbation of the SC lattice that gives homoclinic cycles. More precisely, we show that a $\mathbf{D}_3$-equivariant perturbation of a $\mathbb{O} \oplus \mathbf{Z}_2^c \dot{+} \mathbf{T}^3$-equivariant bifurcation problem can give rise to homoclinic cycles. Section 4 considers similar examples on the FCC lattice, whilst Sec. 5 considers the BCC lattice. In each case our results apply to solutions on these lattices with symmetry isomorphic to $\mathbb{O} \oplus \mathbf{Z}_2^c$, however, they are all limited to the fundamental representation of Γ. In Sec. 5.1 we prove the results for the fundamental representation of a cubic lattice apply, with a few restrictions, to all translation free irreducible representations of the lattice.

Section 6 provides some concluding remarks and discusses relavant physical systems.

A general classification of the heteroclinic behaviour resulting from the perturbations of cubic planforms is underway [Parker, 2006].

2 Problem Formulation

We begin by recalling the main points of forced symmetry-breaking of group orbits of steady states [Lauterbach & Roberts, 1992; Lauterbach et al., 1996]. We also briefly discuss steady-state symmetry-breaking bifurcations of Euclidean-invariant PDEs, see [Golubitsky et al., 1988; Parker, 2003; Dionne et al., 1997].

2.1 Forced symmetry-breaking

Let X be a smooth finite-dimensional manifold. Let Γ be a compact Lie group acting smoothly on X:

$$\Gamma \times X \mapsto X, \quad (\gamma, x) \to \gamma x.$$

Then Lauterbach & Roberts [1992, Proposition 1.1] prove:

Theorem 2.1. *Let Γ be a compact Lie group acting smoothly on a finite-dimensional smooth manifold X. Let f be a Γ-equivariant vector field on X and suppose that Φ_f is the flow on X corresponding to f. Let $X_0 \subset X$ be a compact submanifold, invariant under the flow Φ_f and the action of Γ. Suppose that X_0 is normally hyperbolic. Let $\Delta \subset \Gamma$ be a subgroup of Γ. Let g be a Δ-equivariant vector field on X. Let Φ_g be the flow on X corresponding to g. Suppose that $\|f - g\| < \varepsilon$. Then, if ε is sufficiently small, there exists a unique manifold X_ε near to X_0, invariant under the flow Φ_g. Moreover, there exist a Δ-equivariant diffeomorphism $\Theta : X_0 \to X_\varepsilon$.*

We call Theorem 2.1 the Equivariant Persistence Theorem. This nomenclature is non-standard, although we have used it before to refer to this theorem [Parker et al., 2006a,b; Parker, 2003]. In our applications X_0 is a manifold of solutions to a equivariant bifurcation problem (with compact symmetry group), which generically is a normally hyperbolic manifold, see [Field, 1980].

Let $x \in X$. The *isotropy subgroup of* x is the subgroup of Γ defined by

$$\Sigma_x = \{\gamma \in \Gamma : \gamma x = x\}.$$

The *orbit of* x is the set

$$\Gamma x = \{\gamma x : \gamma \in \Gamma\}.$$

All elements on the same group orbit have isotropy subgroup conjugate to Σ_x. It is well known that Γx is Γ-equivariantly diffeomorphic to the homogeneous space Γ/Σ_x [Golubitsky et al., 1988]. Here $\Gamma/\Sigma_x = \{\gamma\Sigma_x : \gamma \in \Gamma\}$ is the space of left cosets. It is usual to drop the x from Σ_x and write Σ. The *fixed-point subspace of* Σ is

$$\text{Fix}(\Sigma) = \{x \in X : \sigma x = x\}.$$

If $\dim \text{Fix}(\Sigma) = 1$ then Σ is an *axial subgroup* of Γ.

Group actions on Γ/Σ. Let Σ be an isotropy subgroup of Γ. Let Δ be a Lie subgroup of Γ. Define an action of Δ on the space Γ/Σ by

$$\Delta \times (\Gamma/\Sigma) \to (\Gamma/\Sigma), \quad (\delta, \gamma\Sigma) \to \delta\gamma\Sigma.$$

Let x be a point with Σ symmetry, since the orbit $X_0 = \Gamma x$ is Γ-equivariantly diffeomorphic to Γ/Σ, there is an induced action of Δ on the manifold X_0. Suppose g is a Δ-equivariant vector field sufficiently close to f. Then there exists a smooth, Δ-invariant and flow-invariant manifold X_ε close to X_0. Moreover, there exists a Δ-equivariant diffeomorphism $\Theta : X_\varepsilon \to X_0$ [Lauterbach & Roberts, 1992]. The Δ-action on X_0 induces a Δ-action on X_ε. Furthermore, Δ-equivariant vector fields (or flows) on X_0 are the restriction of a Δ-equivariant perturbation of a Γ-equivariant vector field on X [Lauterbach & Roberts, 1992, Proposition 1.3].

Let Δ' be a subgroup of Δ. Then the *fixed-point subset of* Δ' is defined by

$$\mathrm{Fix}_{\Gamma/\Sigma}(\Delta') = \{x \in \Gamma/\Sigma | \delta x = x \text{ for all } \delta \in \Delta'\}.$$

Note that $\mathrm{Fix}_{\Gamma/\Sigma}$ is computed with respect to the Δ action on Γ/Σ rather than on X. The fixed-point subset $\mathrm{Fix}_{\Gamma/\Sigma}(\Delta')$ is invariant under Δ-equivariant flows. Define the *isotropy subgroup of* $x \in \Gamma/\Sigma$ by

$$\mathrm{Stab}(x) = \{\delta \in \Delta | \delta x = x\}.$$

This notation is used to differentiate between the isotropy subgroup of $x \in \Gamma/\Sigma$ with respect to the Δ-action on Γ/Σ, and the isotropy subgroup of $x \in X$ with respect to the Γ action on X. Let $x \in \Gamma/\Sigma$. Let $C = C(x)$

be the connected component of $\mathrm{Fix}(\mathrm{Stab}(x)) \subseteq \Gamma/\Sigma$ which contains x. Let $\mathscr{C}_\Delta$ be the collection of those Cs which are homeomorphic to $\{0\}$ or S^1. If $C \in \mathscr{C}_\Delta$ is homeomorphic to S^1, then we call C a *connecting orbit.* The set $\mathscr{C}_\Delta$ is invariant under the Δ-action.

Let $C \in \mathscr{C}_\Delta$ be a connecting orbit. Choose a parametrization ω for C. We write

$$C := C_\omega.$$

In our work the range of ω is $\mathbf{T}^3$, so ω is determined by three coordinate functions θ_j for $j = 1, 2, 3$. Thus we write

$$C = C_{(\theta_1,\theta_2,\theta_3)}.$$

Definition 2.2. *Let*

$$\mathbb{X}_\Delta = \bigcup_{C \in \mathscr{C}_\Delta} C \subset \Gamma/\Sigma.$$

The set $\mathbb{X}_\Delta$ *is called the* skeleton of Γ/Σ with respect to Δ.

To save cumbersome language we shall use the term *skeleton* when the context is clear. A Δ-equivariant flow on Γ/Σ induces a Δ-equivariant flow on $\mathbb{X}_\Delta$. $\mathbb{X}_\Delta$ is a stratified manifold, in fact the strata are flow-invariant. Let $x \in \mathbb{X}_\Delta$. Let $S(x)$ be the connected component of the stratum containing x. Define $\mathscr{S}_\Delta = \{S(x) | x \in \mathbb{X}_\Delta\}$. Define flow-invariant subsets of $\mathbb{X}_\Delta$ as follows: given the set $\mathscr{S}_\Delta$ define

$$E_{(\Delta,\Gamma/\Sigma)} = \{S \in \mathscr{S}_\Delta | S \text{ is homeomorphic to } \{0\}\},$$
$$H_{(\Delta,\Gamma/\Sigma)} = \{S \in \mathscr{S}_\Delta | S \text{ is homeomorphic to } \mathbb{R}\}.$$

Since Γ/Σ is diffeomorphic to X_0 we also write $E_{(\Delta,X_0)} \equiv E_{(\Delta,\Gamma/\Sigma)}$ and $H_{(\Delta,X_0)} \equiv H_{(\Delta,\Gamma/\Sigma)}$; it is sometimes more convenient to use this notation.

Homoclinic cycles. A *heteroclinic orbit* h between two steady states e_1 and e_2 is a trajectory that is forward asymptotic to e_2 and backward asymptotic to e_1. A *heteroclinic cycle* is an invariant set consist of the union of the steady states $e_1, \ldots, e_n$ where the indices are taken modulo n; that is, $e_{n+1} = e_1$. If $n = 1$ then we have a homoclinic cycle. [1]

Symmetry properties of the skeleton. Let $C \in \mathscr{C}_\Delta$ or $\mathscr{S}_\Delta$. The *pointwise isotropy of* C is defined by

$$\mathrm{stab}(C) = \{\delta \in \Delta | \delta x = x \text{ for all } x \in C\}.$$

There is an induced action of Δ on $\mathscr{C}_\Delta$ given by permutation. The *setwise isotropy of* $C \in \mathscr{C}_\Delta$ is

$$\mathrm{Stab}(C) = \{\delta \in \Delta | \delta C = C\}.$$

[1] Better definition.

The pointwise isotropy, $\mathrm{stab}(C)$, of $C \in \mathscr{C}_\Delta$ is normal in the setwise isotropy $\mathrm{Stab}(C)$. Thus we can form the quotient group:

$$\mathrm{Stab}(C)/\mathrm{stab}(C).$$

This group has a natural action on C.

Let $C \in \mathscr{C}_\Delta$ be a connecting orbit and $x \in C$. Suppose that x is a fixed-point of some element of $\mathrm{Stab}(C)$, then x is a *knot relative to* C. Note that a knot must be an element of $E_{(\Delta,\Gamma/\Sigma)}$, however the converse is clearly false, it can be false even for those steady states that lie on connecting orbits [Parker et al., 2006b]. Lauterbach et al. [1996] prove the following:

Lemma 2.3. *Given* $e \in E_{(\Delta,\Gamma/\Sigma)}$ *or* $h \in H_{(\Delta,\Gamma/\Sigma)}$. *Then*

$$\mathrm{Stab}(e)/\mathrm{stab}(e) = \mathrm{Stab}(h)/\mathrm{stab}(h) = \mathbf{1}$$

the trivial group.

Let $H_{(\Delta,\Gamma/\Sigma)}(C)$ denote the set $\{h \in H_{(\Delta,\Gamma/\Sigma)} | h \subset C\}$. Lauterbach et al. [1996] give the following general symmetry constraints on $\mathscr{C}_\Delta$.

Proposition 2.4. *Let* $C \in \mathscr{C}_\Delta$ *with* $H_{(\Delta,\Gamma/\Sigma)}(C) \neq \emptyset$. *Suppose* $\mathrm{Stab}(C)$ *contains* $m \in \mathbb{N}$ *orientation reversing elements and*

$$\mathrm{Stab}(C)/\mathrm{stab}(C) = \mathbf{D}_m,$$

where the convention $\mathbf{D}_1 = \mathbf{Z}_2$ *is employed. The group* $\mathrm{Stab}(C)/\mathrm{stab}(C)$ *acts as the group of* m *reflections about axes through opposite knots and* $m-1$ *nontrivial rotations.*

Suppose $\mathrm{Stab}(C)$ *contains no orientation reversing elements; then*

$$\mathrm{Stab}(C)/\mathrm{stab}(C) = \mathbf{Z}_m$$

for some $m \in \mathbb{N}$. *Where we employ the convention* $\mathbf{Z}_1 = \mathbf{1}$. *The group* $\mathrm{Stab}(C)/\mathrm{stab}(C)$ *acts as rotations on* C.

The rotations and reflections preserve $H_{(\Delta,\Gamma/\Sigma)}(C)$.

If $H_{(\Delta,\Gamma/\Sigma)}(C) \neq \emptyset$, then the knots relative to C always occur in pairs on $\mathbb{X}_\Delta$ dividing C into two connected components with the same number of edges in $H_{(\Delta,\Gamma/\Sigma)}(C)$ on each component. There are two possible types of behaviour [Lauterbach et al., 1996]:

Proposition 2.5. *Suppose there are no knots relative to* C. *Then*

$$\mathrm{Stab}(C)/\mathrm{stab}(C)$$

is isomorphic to some $\mathbf{Z}_m$ *(with* $\mathbf{Z}_1 = \mathbf{1}$*) and acts on* $H_{(\Delta,\Gamma/\Sigma)}(C)$ *by rotations on* C.

Suppose that there are knots relative to C. *Then* $\mathrm{Stab}(C)/\mathrm{stab}(C)$ *is isomorphic to some* $\mathbf{D}_m$ *(with* $\mathbf{D}_1 = \mathbf{Z}_2$*). This group acts as* $m-1$ *non-*

trivial rotations and m reflections on $H_{(\Delta,\Gamma/\Sigma)}(C)$. *The pairs of opposite knots give the axes of reflections.*

Vector fields on the skeleton. Define

$$\pi : \mathbb{X}_\Delta \to \Delta\backslash\mathbb{X}_\Delta \quad \text{by} \quad \pi(x) = {}_\Delta[x],$$

where ${}_\Delta[x]$ is the equivalence class of all points x which are members of the same group orbit. Let $\mathbb{X}^p_\Delta = \Delta\backslash\mathbb{X}_\Delta$. Then $\mathbb{X}^p_\Delta$ is called the *projected skeleton.* By the smooth lifting theorem of Schwartz (see [Lauterbach & Roberts, 1992]), π is surjective. So every stratum preserving smooth vector field on $\Gamma\backslash\mathbb{X}_\Delta$ lifts to a smooth Γ-equivariant vector field on $\mathbb{X}_\Delta$, and any flow on the projected skeleton lifts to a Δ-equivariant flow on the skeleton. Lauterbach et al. [1996, Corollary 3.32] prove:

Theorem 2.6. *Let* $h \in H_{(\Delta,\Gamma/\Sigma)}$. *Then there exist* Δ*-equivariant vector fields and corresponding flows* Φ *on* $\mathbb{X}_\Delta$ *such that* h *is a heteroclinic orbit of* Φ *connecting steady states in* $E_{(\Delta,\Gamma/\Sigma)}$.

In particular, loops on the projected skeleton give homoclinic cycles on the skeleton.

Let $C \in \mathscr{C}_\Delta$, then $\overline{C} := \Delta\backslash C$; that is, $\overline{C}$ denotes the projection of C into the Δ-orbit space.

The projected skeleton classifies all distinct Δ-orbits of steady states and heteroclinic orbits that are forced to exist by the residual Δ symmetry.

Perturbed flows. The manifold X_0 is only a model for the true perturbed flow-invariant manifold X_ε. In certain cases there is a precise relation between the qualitative behaviour of flows on X_0 and X_ε. For further details see [Parker et al., 2006a,b; Parker, 2003].

Let $\Theta_\varepsilon : X_0 \to X_\varepsilon$ be the Δ-equivariant diffeomorphism given in Theorem 2.1. Then Θ_0 is the identity map.

Theorem 2.7. *Suppose* $e_1, e_2 \in E_{(\Delta,X_0)}$ *and* $h \in H_{(\Delta,X_0)}$ *is such that* $\partial h = \{e_1, e_2\}$. *Suppose the flow along* h *has no steady states. Then, provided* ε *is sufficiently small,* $\Theta_\varepsilon(h)$ *satisfies: 1)* $\partial\Theta_\varepsilon(h) = \{\Theta_\varepsilon(e_1), \Theta_\varepsilon(e_2)\}$ *and 2) there are no steady states for the perturbed flow along* $\Theta_\varepsilon(h)$.

Proof. This is a simple perturbation argument, the details of which can be found in [Parker, 2003; Maier-Paape & Lauterbach, 2000] □

The following is immediate:

Corollary 2.8. *Suppose the projective skeleton contains a topological circle with a single steady-state, then the perturb flow has a homoclinic cycle.*

Proof. Theorem 2.6 guarantees the existence of flows on the connection without steady states, then apply Theorem 2.7. □

2.2 Partial differential equations with Euclidean symmetry

Consider a parameterized family of PDEs

$$\frac{\partial}{\partial t}u(x,t) = F\left(u(x,t),\lambda\right) \tag{2}$$

where $F : \mathcal{X} \times \mathbb{R} \to \mathcal{Y}$ is a nonlinear operator between suitable function spaces $\mathcal{X}$ and $\mathcal{Y}$, and $\lambda \in \mathbb{R}$ is a bifurcation parameter. The function $u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^n$ is a function in $\mathcal{X}$ of a spatial variable $x \in \mathbb{R}^3$ and time t.

Assume Eq. (2) is $\mathbf{E}(3)$-equivariant, where $\mathbf{E}(3)$ is the three-dimensional Euclidean group. In addition, we assume that there is an Euclidean-invariant time-independent solution of Eq. (2) for all values of λ. Without loss of generality we assume that this spatially uniform solution corresponds to $u = 0$, that is

$$F(0,\lambda) = 0,$$

for all λ. Furthermore, we assume that this solution is stable for $\lambda < 0$, unstable for $\lambda > 0$ and that $\lambda = 0$ corresponds to a steady-state symmetry-breaking bifurcation point.

We seek spatially periodic, time-independent solutions $u(x,t)$ to Eq. (2). A function u is $\mathcal{L}$-periodic if $u(x+\ell) = u(x)$ for all $\ell \in \mathcal{L}$. The subspace $\mathcal{X}_{\mathcal{L}} \subset \mathcal{X}$ of $\mathcal{L}$-periodic functions is $\mathcal{X}_{\mathcal{L}} = \{f \in \mathcal{X} | f(x+\ell) = f(x) \text{ for all } \ell \in \mathcal{L}\}$. The group Γ is the largest subgroup of $\mathbf{E}(3)$ that preserves $\mathcal{X}_{\mathcal{L}}$; that is, $\gamma \mathcal{X}_{\mathcal{L}} \subseteq \mathcal{X}_{\mathcal{L}}$ for all $\gamma \in \Gamma$.

The dual lattice $\mathcal{L}^*$ of $\mathcal{L}$ is the set

$$\mathcal{L}^* = \left\{ \mathbf{k} \in \mathbb{R}^3 | x \mapsto e^{2\pi \mathrm{i} \mathbf{k} \cdot x} \text{ is } \mathcal{L}\text{−periodic} \right\}.$$

We assume that a function $u \in \mathcal{X}_{\mathcal{L}}$ can be written in the form

$$u(x,t) = \sum_{j=1}^{s} z_j e^{2\pi \mathrm{i} \mathbf{K}_j \cdot x} + c.c., \tag{3}$$

where $z_j \in \mathbb{C}$ and *c.c.* denotes complex conjugate. The sphere $|\mathbf{K}_j| = k_c$ in the three-dimensional $\mathbf{k}$-space is called the *critical sphere.* If the wavelength of the instabilities coincides with the periodicity of the functions in $\mathcal{X}_{\mathcal{L}}$, then we call this the *fundamental representation of the lattice.* The dimension of the bifurcation problem depends on the number of vectors $\mathbf{k} \in \mathcal{L}^*$ which lie on the critical sphere. The PDEs, by a Liapunov–Schmidt reduction [Golubitsky & Schaeffer, 1985] or restriction to the centre manifold [Carr, 1981], give a system of ODEs on $\mathbb{C}^s$. The representation of Γ on $\mathbb{C}^s$ is determined by its action on the complex amplitudes z_j in Eq. (3). The reduction gives a system of ODEs

$$\dot{z} = f(z, \lambda),$$

where $f : \mathbb{C}^s \times \mathbb{R} \to \mathbb{C}^s$ is Γ-equivariant, $f(0, \lambda) = 0$ and the Jacobian matrix at the bifurcation point $(\mathrm{d}f)_{0,0}$, is the zero matrix. Using standard equivariant bifurcation theory [Golubitsky et al., 1988] we determine the axial subgroup for this system. We call an axial subgroups Σ *translation free* if

$$\Sigma \cap \mathbf{T}^n = \mathbf{1}.$$

This paper considers the three cubic lattices. Let $\mathcal{L}$ be a cubic lattice. The symmetry group of $\mathcal{L}$ is $\Gamma = \mathbb{O} \oplus \mathbf{Z}_2^c \dot{+} \mathbf{T}^3$. Here $\mathbb{O}$ is the orientation preserving symmetries of the cube, $\mathbf{Z}_2^c$ is inversion through the origin and $\mathbf{T}^3$ is the group of translations modulo $\mathcal{L}$. We use $\dot{+}$ to denote the semi-direct product. Given $(\gamma_1, \theta_1), (\gamma_2, \theta_2) \in \Gamma$ the product is defined to be

$$(\gamma_1, \theta_1)(\gamma_2, \theta_2) = (\gamma_1\gamma_2, \gamma_1\theta_2 + \theta_1).$$

3 Simple Cubic Lattice

The wave vector that generate the fundamental representation of the simple cubic (SC) are:

$$\mathbf{K}_1 = (1,0,0), \quad \mathbf{K}_2 = (0,1,0), \quad \mathbf{K}_3 = (0,0,1),$$

see [Dionne, 1993]. The action of $\Gamma = \mathbb{O} \oplus \mathbf{Z}_2^c \dot{+} \mathbf{T}^3$ on $\mathbb{C}^3$ is generated as follows: choose coordinates $z = (z_1, z_2, z_3)$ on $\mathbb{C}^3$, then Γ acts by

$$\begin{aligned}
\rho_x(z) &= (z_1, \overline{z_3}, z_2), \\
\rho_y(z) &= (z_3, z_2, z_1), \\
c(z) &= (\overline{z_1}, \overline{z_2}, \overline{z_3}), \\
\theta(z) &= (e^{\mathrm{i}\theta_1} z_1, e^{\mathrm{i}\theta_2} z_2, e^{\mathrm{i}\theta_3} z_3).
\end{aligned}$$

Here ρ_x and ρ_y are the generators of $\mathbb{O}$, c generates the group $\mathbf{Z}_2^c$, and $\theta \in \mathbf{T}^3$. The Γ-action on $\mathbb{C}^3$ has seven conjugacy classes of isotropy subgroups [Callahan & Knobloch, 1997], only three are axial, of these one is translation free, namely $\mathbb{O} \oplus \mathbf{Z}_2^c$ [Dionne, 1993; Callahan & Knobloch, 1997]. We call this the SC solution. Callahan & Knobloch [1997] show that if certain nondegeneracy conditions hold then the bifurcation problem is fully determined by the third-order truncation of a general Γ-equivariant vector field. In particular, the authors show that the SC solution can be stable at bifurcation, see [Callahan & Knobloch, 1997] for the details the stability conditions. The group orbit of the SC solution is a 3-torus, which we denote throughout by X_0. The coordinate system on X_0 is $\theta = (\theta_1, \theta_2, \theta_3)$, where $\theta_j \in [0, 2\pi)$. The manifold X_0 is, generically, normally hyperbolic [Field, 1980].

Let $\mathbf{D}_3$ be the subgroup of Γ generated by

$$\begin{aligned} \tau z &= (\overline{z_3}, z_1, \overline{z_2}), \\ \kappa z &= (z_3, \overline{z_2}, z_1). \end{aligned}$$

To understand this group consider a cube and fix a arc joining two opposite vertices through the centre of the cube. The subgroup of $\mathbb{O}$ that fixes this line is given by (a conjugate of) $\mathbf{D}_3$. Indeed, there are four lines giving four conjugate copies of $\mathbf{D}_3$. The results for the three other $\mathbf{D}_3$ subgroups are isomorphic to the results derived below.

We consider a $\mathbf{D}_3$-equivariant perturbation of a general Γ-equivariant bifurcation problem on the SC lattice. Since X_0 is normally hyperbolic the Equivariant Persistence Theorem guarantees that if the perturbation is sufficiently small there exists a flow-invariant manifold X_ε that is $\mathbf{D}_3$-equivariantly diffeomorphic to X_0.

There is an induced $\mathbf{D}_3[\tau,\kappa]$-action on X_0 generated by

$$\begin{aligned}\tau(\theta_1,\theta_2,\theta_3) &= (-\theta_3,\theta_1,-\theta_2),\\ \kappa(\theta_1,\theta_2,\theta_3) &= (\theta_3,-\theta_2,\theta_1).\end{aligned}$$

This action implies that:

$$\begin{aligned}\mathscr{C}_{\mathbf{D}_3[\tau,\kappa]} &= \{(0,0,0),(\pi,\pi,\pi),C_{(\theta,-\theta,0)},C_{(\theta,-\theta,\pi)},C_{(\theta,0,\theta)},\\ &\quad C_{(\theta,\pi,\theta)},C_{(0,\theta,\theta)},C_{(\pi,\theta,\theta)},C_{(\theta,\theta,-\theta)}\},\end{aligned}$$

The $\mathbf{D}_3[\tau,\kappa]$-action on X_0 induces an action on $\mathscr{C}_{\mathbf{D}_3[\tau,\kappa]}$. This computation is straight forward and in Table 1 we list the elements of $\mathbf{D}_3[\tau,\kappa]$ that act nontrivially on $\mathscr{C}_{\mathbf{D}_3[\tau,\kappa]}$. A simple computation using The $\mathbf{D}_3[\tau,\kappa]$-actions on X_0 and on $\mathscr{C}_{\mathbf{D}_3[\tau,\kappa]}$ verifies the isotropy data in Table 2. The isotropy data in Table 2 shows that $\mathrm{Stab}(C)/\mathrm{stab}(C)$ is trivial unless $C = C_{(\theta,\theta,-\theta)}$ with

$$\mathrm{Stab}(C)/\mathrm{stab}(C) \cong \mathbf{Z}_2$$

where $\mathbf{Z}_2$ act as a reflection on the connecting orbit. Thus there is an axis of symmetry along $C_{(\theta,\theta,-\theta)}$. The only steady states that lie on $C_{(\theta,\theta,-\theta)}$ are $(0,0,0)$ and (π,π,π). This implies these two steady states are the knots relative to $C_{(\theta,\theta,-\theta)}$.

The group action in Table 1 implies there are three group orbit representative for the connecting orbits: $\overline{C}_{(\theta,-\theta,0)}$, $\overline{C}_{(\theta,-\theta,\pi)}$ and $\overline{C}_{(\theta,\theta,-\theta)}$. The only steady states on the projected skeletons are $\overline{(0,0,0)}$ and $\overline{(\pi,\pi,\pi)}$. Since the connection $C_{(\theta,\theta,-\theta)}$ has two knots it projects into the orbit space as a arc joining the two knots. More precisely, we have the following relations: $\overline{C}_{(\theta,-\theta,0)}$ connects $\overline{(0,0,0)}$ to itself, $\overline{C}_{(\theta,-\theta,\pi)}$ connects $\overline{(\pi,\pi,\pi)}$ to itself and $\overline{C}_{(\theta,\theta,-\theta)}$ connects $\overline{(0,0,0)}$ to $\overline{(\pi,\pi,\pi)}$. Figure 1 illustrates the projected skeleton.

Theorem 3.1. *Let f be an Γ-equivariant bifurcation problem on the SC lattice. Let X_0 be the group orbit of SC solutions. Then there exists a $\mathbf{D}_3[\tau,\kappa]$-equivariant perturbation of f such that the flow on the perturbed group orbit of X_0 has a homoclinic cycle.*

Proof. The projected skeleton supports homooclinic cycles provided that there are no additional steady states along $\overline{C}_{(\theta,-\theta,0)}$ and $\overline{C}_{(\theta,-\theta,\pi)}$. Theorem 2.6 guarantees the existence of such perturbations. $\square$

4 Face Centred Cubic Lattice

We consider the fundamental representation of the FCC lattice. The wave vectors in this case are given by [Dionne et al., 1997]:

$$\mathbf{K}_1 = (1,1,1), \quad \mathbf{K}_2 = (1,-1,1), \quad \mathbf{K}_3 = (1,-1,-1), \quad \mathbf{K}_4 = (1,1,-1).$$

The representation of Γ on $\mathbb{C}^4$ corresponds to the following action. Choose coordinates $z = (z_1, z_2, z_3, z_4)$ on $\mathbb{C}^4$, then Γ-action is generated by

$$\begin{aligned}
\rho_x(z) &= (z_4, z_1, z_2, z_3), \\
\rho_y(z) &= (\overline{z_3}, \overline{z_4}, z_2, z_1), \\
c(z) &= (\overline{z_1}, \overline{z_2}, \overline{z_3}, \overline{z_4}), \\
\theta(z) &= (e^{-\mathrm{i}(\theta_1+\theta_3)} z_1, e^{\mathrm{i}\theta_2} z_2, e^{\mathrm{i}(\theta_2+\theta_3)} z_3, e^{-\mathrm{i}\theta_1} z_4).
\end{aligned}$$

The notation used is the same as in the SC case. The Γ-action on $\mathbb{C}^4$ has fifteen conjugacy classes of isotropy subgroups, only four are axial, and there are two conjugacy classes of translation free axial subgroup given by $\mathbb{O} \oplus \mathbf{Z}_2^c$ and $\widetilde{\mathbb{O}} \oplus \mathbf{Z}_2^c$: the FCC and double-diamond solutions, respectively [Callahan & Knobloch, 1997; Dionne et al., 1997]. We consider only these two solutions.

Callahan & Knobloch [1997] show that if certain nondegeneracy conditions hold then the bifurcation problem is fully determined by the third-order truncation of a general Γ-equivariant vector field. Indeed, they compute a normal form and classify the bifurcation diagrams. The authors show the FCC and double-diamond solutions can be stable at bifurcation, but not simultaneously. The group orbit of FCC and double-diamond solutions are 3-tori and, generically, are normally hyperbolic [Field, 1980]. We consider the FCC solution with group orbit X_0 and later show these results apply directly to the double-diamond solution.

Unlike the SC case we give three different perturbation that give rise to homoclinic cycles. The symmetries of the perturbations are: $\mathbf{D}_4[\rho, \kappa_1]$, $\mathbb{T}[\tau, \rho_1]$ and $\mathbf{D}_3[\tau, \kappa]$. The $\mathbf{D}_3[\tau, \kappa]$ subgroup is identical to the group in the SC case. The actions of these groups are generated as follows. The group $\mathbf{D}_4[\rho, \kappa_1]$ is generated by:

$$\begin{aligned} \rho z &= (z_4, z_1, z_2, z_3), \\ \kappa_1 z &= (\overline{z_1}, \overline{z_4}, \overline{z_3}, \overline{z_2}). \end{aligned}$$

The group $\mathbb{T}[\tau, \rho_1]$ is the symmetry group of a tetrahedron and is generated by

$$\tau z = (\overline{z_2}, \overline{z_3}, z_1, z_4)$$

and $\rho_1 = \rho^2$. Finally, the action of $\mathbf{D}_3[\tau, \kappa]$ is generated by τ and

$$\kappa = (z_2, z_1, \overline{z_3}, \overline{z_4}).$$

The group $\mathbf{D}_4[\rho, \kappa_1]$ is the subgroup of $\mathbb{O}$ that fixes an axes through the midpoint of two opposite faces through the origin. There are three conjugate subgroups of $\mathbb{O}$ with this property corresponding to the three mutually perpendicular axes (through opposite faces) in the cube. The results for the other two $\mathbf{D}_4$ subgroups are isomorphic to the results we derive for $\mathbf{D}_4[\rho, \kappa_1]$. The group $\mathbb{T}[\tau, \rho_1]$ is the subgroup of $\mathbb{O}$ corresponding to the symmetry group of a tetrahedron. There is only one conjugacy class of this subgroup in $\mathbb{O}$ ($\mathbb{T}$ has index 2 in $\mathbb{O}$). The subgroup $\mathbf{D}_3$ was discussed above and similar comments remain true.

We examine the effect on the FCC group orbit when perturbation terms with $\mathbf{D}_4[\rho, \kappa_1]$, $\mathbb{T}[\tau, \rho_1]$ and $\mathbf{D}_3[\tau, \kappa]$ symmetry are added to a general Γ-equivariant bifurcation problem. Below we show these result apply to the double-diamond solutions with only minimal changes. In each case the Equivariant Persistence Theorem guarantees the existence of a flow-invariant manifold X_ε that is diffeomorphic to X_0, where the diffeomorphism is equivariant with respect to the symmetry of the perturbation.

The groups listed above have induced actions on X_0, these are generated by:

$$
\begin{aligned}
\rho(\theta_1,\theta_2,\theta_3) &= (-(\theta_2+\theta_3), -(\theta_1+\theta_3), \theta_1+\theta_2+\theta_3)\,, \\
\kappa_1(\theta_1,\theta_2,\theta_3) &= (\theta_1+\theta_3, -\theta_2, -\theta_3)\,, \\
\tau(\theta_1,\theta_2,\theta_3) &= (\theta_1, -(\theta_2+\theta_3), \theta_2-\theta_1) \\
\kappa(\theta_1,\theta_2,\theta_3) &= (-\theta_1, -(\theta_1+\theta_3), \theta_1-\theta_2)
\end{aligned}
$$

Verification of these actions is straightforward. The induced actions on X_0 imply:

$$
\begin{aligned}
\mathscr{C}_{\mathbf{D}_4[\rho,\kappa_1]} &= \{(0,0,0),(0,0,\pi),(\pi,\pi,0),(\pi,\pi,\pi),(3\pi/2,\pi/2,\pi),(\pi/2,3\pi/2,\pi), \\
&\quad C_{(0,0,\theta)}, C_{(\pi,\pi,\theta)}, C_{(\theta,\theta,0)}, C_{(\theta,-\theta,0)}, C_{(\theta,-\theta,\pi)}, C_{(\theta,\theta,-\theta)}, C_{(\theta,\theta,-2\theta)}, \\
&\quad C_{(\theta,\theta,\pi-\theta)}, C_{(\theta+\pi,\theta,\pi)}, C_{(\theta+\pi,\theta,-2\theta)}\} \\
\mathscr{C}_{\mathbb{T}[\tau,\rho_1]} &= \{(0,0,0),(\pi,\pi,0),(3\pi/2,\pi/2,\pi),(\pi/2,3\pi/2,\pi), C_{(\theta,\theta,0)}, C_{(\theta,-\theta,0)}, \\
&\quad C_{(\theta,-\theta,\pi)}, C_{(\theta,-\theta,2\theta)}, C_{(\theta,-\theta,-2\theta)}, C_{(3\theta,\theta,-2\theta)}, C_{(\theta,3\theta,-2\theta)}, C_{(\theta,\theta,-2\theta)}, \\
&\quad C_{(\theta+\pi,\theta,\pi)}, C_{(\theta+\pi,\theta,-2\theta)}\}. \\
\mathscr{C}_{\mathbf{D}_3[\tau,\kappa]} &= \{(0,0,0),(\pi,\pi,0), C_{(0,\theta,0)}, C_{(\pi,\theta,0)}, C_{(3\theta,\theta,-2\theta)}, C_{(0,\theta,-\theta)}, C_{(\pi,\theta,\pi-\theta)}, \\
&\quad C_{(0,0,\theta)}, C_{(\pi,\pi,\theta)}\}. \qquad (4)
\end{aligned}
$$

The action of $\mathbf{D}_4[\rho,\kappa_1]$, $\mathbb{T}[\tau,\rho_1]$ and $\mathbf{D}_3[\tau,\kappa]$ on X_0 all induce actions on the sets $\mathscr{C}_{\mathbf{D}_4[\rho,\kappa_1]}$, $\mathscr{C}_{\mathbb{T}[\tau,\rho_1]}$ and $\mathscr{C}_{\mathbf{D}_3[\tau,\kappa]}$ respectively. To describe the action of

$\mathbb{T}[\tau, \rho_1]$ on $\mathscr{C}_{\mathbb{T}[\tau,\rho_1]}$ it is useful to introduce some additional notation. We let $\tau_1 = \tau\rho^2$, $\tau_2 = \tau^2\rho^2$, $\tau_3 = \rho^2\tau^2\rho^2$, $\rho_2 = \tau_3\tau$ and $\rho_3 = \tau\tau_3$. The elements ρ_j, $j = 1, 2, 3$ have order 2 and the elements τ_j, $j = 1, 2, 3$ have order 3. This notations implies that

$$\mathbb{T}[\tau, \rho_1] = \{e, \rho_1, \rho_2, \rho_3, \tau, \tau^2, \tau_1, \tau_1^2, \tau_2, \tau_2^3, \tau_3, \tau_3^2\}$$

where e denotes the identity element. The actions of $\mathbf{D}_4[\rho, \kappa_1]$, $\mathbb{T}[\tau, \rho_1]$ and $\mathbf{D}_3[\tau, \kappa]$ on X_0 all induce actions on the sets $\mathscr{C}_{\mathbf{D}_4[\rho,\kappa_1]}$, $\mathscr{C}_{\mathbb{T}[\tau,\rho_1]}$ and $\mathscr{C}_{\mathbf{D}_3[\tau,\kappa]}$ respectively are contained in Tables 3–5. In each case only those elements acting nontrivially are shown. Using the group actions on X_0 and the entries from Tables 3–5 we can derive the pointwise and setwise isotropy subgroups. This isotropy data is contained in Tables 6–8.

The $\mathbf{D}_4[\rho, \kappa_1]$ case. The $\mathbf{D}_4[\rho, \kappa_1]$-action on $\mathscr{C}_{\mathbf{D}_4[\rho,\kappa_1]}$ in Table 3 implies there are six group orbit representative for the connecting orbits in $\mathscr{C}_{\mathbf{D}_4[\rho,\kappa_1]}$. We take

$$\overline{C}_{(0,0,\theta)}, \overline{C}_{(\pi,\pi,\theta)}, \overline{C}_{(\theta,\theta,0)}, \overline{C}_{(\theta,-\theta,0)}, \overline{C}_{(\theta,-\theta,\pi)}, \overline{C}_{(\theta+\pi,\theta,\pi)}.$$

The representatives for the steady states are $\overline{(0,0,0)}$, $\overline{(0,0,\pi)}$, $\overline{(\pi,\pi,0)}$, and $\overline{(3\pi/2,\pi/2,\pi)}$. Each connecting orbit C satisfies

$$\mathrm{Stab}(C)/\mathrm{stab}(C) \cong \mathbf{Z}_2,$$

where $\mathbf{Z}_2$ acts as a reflection on the connecting orbit. This implies that each connecting orbit has two knots. The connection $C_{(\theta+\pi,\theta,\pi)}$ is different from the other connecting orbits. The two knots are $(3\pi/2, \pi/2, \pi)$ and $(\pi/2, 3\pi/2, \pi)$, but since these points lie in the same group orbit, they project into the orbit space as one point. Thus the group orbit representative $\overline{C}_{(\theta+\pi,\theta,\pi)}$ connects the representative $\overline{(3\pi/2, \pi/2, \pi)}$ to itself. The $\mathbf{D}_4[\rho, \kappa_1]$-action means it is sufficient to determine the knots relative to the group orbit representatives listed above. This information is summarized in Table 9.

We now compute the projected skeleton. The orbit representatives for the steady states are $\overline{(0,0,0)}$, $\overline{(0,0,\pi)}$, $\overline{(\pi,\pi,0)}$, and $\overline{(3\pi/2, \pi/2, \pi)}$. As discussed above each connecting orbit has two knots, this implies that each connecting orbits projects into the orbit space as a arc joining the two knots. The precise relations are as follows:

1. $\overline{C}_{(0,0,\theta)}$ connects $\overline{(0,0,0)}$ to $\overline{(0,0,\pi)}$,
2. $\overline{C}_{(\pi,\pi,\theta)}$ connects $\overline{(0,0,\pi)}$ to $\overline{(\pi,\pi,0)}$,
3. $\overline{C}_{(\theta,\theta,0)}$ connects $\overline{(0,0,0)}$ to $\overline{(\pi,\pi,0)}$,
4. $\overline{C}_{(\theta,-\theta,0)}$ connects $\overline{(0,0,0)}$ to $\overline{(\pi,\pi,0)}$,
5. $\overline{C}_{(\theta,-\theta,\pi)}$ connects $\overline{(0,0,\pi)}$ to $\overline{(3\pi/2, \pi/2, \pi)}$,

6. $\overline{C}_{(\theta+\pi,\theta,\pi)}$ connects $\overline{(3\pi/2,\pi/2,\pi)}$ to itself.

In Fig. 2(a) we illustrate the projected skeleton. Since the projected skeleton contains a loop we immediately deduce the following result.

Theorem 4.1. *Let f be a $\mathbb{O}\oplus\mathbf{Z}_2^c\dot{+}\mathbf{T}^3$-equivariant bifurcation problem. Let X_0 be the group orbit of FCC solutions. Then there exists a small $\mathbf{D}_4[\rho,\kappa_1]$-equivariant perturbations of f such that there exists a persistent steady state on X_0 and a homoclinic cycle to this steady state.*

Proof. The follows from Corollary 2.8. □

The projected skeleton can potentially support dynamics considerably more complex than homoclinic cycles. For example, the projected skeleton can support a heteroclinic cycle between the steady states $\overline{(0,0,\pi)}$, $\overline{(\pi,\pi,0)}$ and $\overline{(0,0,0)}$. Proving the existence of this cycle requires knowledge of the $\mathbf{D}_4[\rho,\kappa_1]$-invariants and -equivariants. Using these we can attempt to classify the flows for different $\mathbf{D}_4[\rho,\kappa_1]$-equivariant perturbations, along the lines given in [Parker et al., 2006a,b]. These computations would be long and involved and are not pursued here.

The $\mathbb{T}[\tau,\rho_1]$ case. Our analysis of the $\mathbb{T}[\tau,\rho_1]$ case follows identical lines to those presented for the $\mathbf{D}_4[\rho,\kappa_1]$ case and we shall be brief.

The entries of Table 4 imply there are three orbit representatives for the connecting orbits:

$$\overline{C}_{(\theta,\theta,0)}, \overline{C}_{(\theta,-\theta,\pi)}, \overline{C}_{(\theta,-\theta,2\theta)}.$$

The representatives of the steady states are: $\overline{(0,0,0)}$, $\overline{(\pi,\pi,0)}$, and $\overline{(3\pi/2,\pi/2,\pi)}$. The orbit representative $\overline{C}_{(\theta,-\theta,2\theta)}$ has no knots, since the quotient group

$$\mathrm{Stab}(C_{(\theta,-\theta,2\theta)})/\mathrm{stab}(C_{(\theta,-\theta,2\theta)})$$

is trivial. The computations of the knots relative to the orbit representatives $\overline{C}_{(\theta,\theta,0)}$, $\overline{C}_{(\theta,-\theta,\pi)}$ are straightforward since each contain two steady states. In each case we compute the knots relative to the representatives $\overline{C}_{(\theta,\theta,0)}$, $\overline{C}_{(\theta,-\theta,\pi)}$ since the remainder follow by symmetry. Following the standard approach we find that the $\mathbb{T}[\tau,\rho_1]$-orbit representatives for the connecting orbits have the following connectivity properties:

1. $\overline{C}_{(\theta,\theta,0)}$ connects $\overline{(0,0,0)}$ to $\overline{(\pi,\pi,0)}$,

2. $\overline{C}_{(\theta,-\theta,2\theta)}$ connects $\overline{(0,0,0)}$ to $\overline{(3\pi/2,\pi/2,\pi)}$ then to $\overline{(\pi,\pi,0)}$, back to $\overline{(3\pi/2,\pi/2,\pi)}$ and finally returns to $\overline{(0,0,0)}$,

3. $\overline{C}_{(\theta,-\theta,\pi)}$ connects $\overline{(3\pi/2,\pi/2,\pi)}$ to itself.

The properties of the final connection follow since $\overline{(\pi/2, 3\pi/2, \pi)} = \overline{(3\pi/2, \pi/2, \pi)}$. Figure 2(b) illustrates the projected skeleton.

Theorem 4.2. *Let f be a $\mathbb{O} \oplus \mathbf{Z}_2^c \dotplus \mathbf{T}^3$-equivariant bifurcation problem on the FCC lattice. Let X_0 be the group orbit of FCC solutions. Then there exists a small $\mathbb{T}[\tau, \rho_1]$-equivariant perturbations of f such that there exists a persistent steady state on the perturbed group orbit of X_0 and homoclinic cycle to this steady state.*

Proof. The follows from Corollary 2.8. □

Like the $\mathbf{D}_4[\rho, \kappa_1]$ case the projected skeleton allows more general dynamics than homoclinic cycles. However, an investigation into these possibilities is beyond the scope of this paper.

The $\mathbf{D}_3[\tau, \kappa]$ case. Our final example follows identical lines to those illustrated above and our description will be brief. The isotropy data in Table 5 implies that only one connecting orbit possesses knots. More precisely, the connecting orbit $C_{(3\theta,\theta,-2\theta)}$ has two knots given by the steady states $(0, 0, 0)$ and $(\pi, \pi, 0)$. The $\mathbf{D}_3[\tau, \kappa]$-orbit representatives for the connecting orbits are $\overline{C}_{(0,\theta,0)}$, $\overline{C}_{(\pi,\theta,0)}$, and $\overline{C}_{(3\theta,\theta,-2\theta)}$. The orbit representatives for the steady states are $\overline{(0,0,0)}$ and $\overline{(\pi,\pi,0)}$. It is now a routine verification to derive the projected skeleton shown in Fig. 2(c).

Theorem 4.3. *Let f be a $\mathbb{O}\oplus\mathbf{Z}_2^c\dot{+}\mathbf{T}^3$-equivariant bifurcation problem on the FCC lattice. Let X_0 be the group orbit of FCC solutions. Then there exists a small $\mathbf{D}_3[\tau_1,\kappa_5]$-equivariant perturbations of f such that there exist two persistent steady states on the perturbed group orbit and homoclinic cycles to these steady states.*

Proof. The follows from Corollary 2.8. □

Double-Diamond solutions. The results derived above for the FCC solution remain true, with some minor changes, if we replace the group orbit of FCC solutions, with the group orbit of double-diamond solutions. We provide direct verification of this claim, subsequently Lemma 5.3 provides a general argument that is representation independent. The isotropy subgroup $\widetilde{\mathbb{O}}\oplus\mathbf{Z}_2^c$ of the double-diamond solutions is generated by $(\rho_x,(0,1/2,1/2))$ and $(\rho_y,(1/2,0,0))$. A simple computation shows that

$$\mathrm{Fix}(\widetilde{\mathbb{O}}\oplus\mathbf{Z}_2^c)=\{(-x,x,x,x)\}$$

where x is non-zero and real. It is easily verified that the induced $\widetilde{\mathbb{O}} \oplus \mathbf{Z}_2^c$-action on X_0 is isomorphic to the $\mathbb{O}\oplus\mathbf{Z}_2^c$-action on X_0. That is, the $\widetilde{\mathbb{O}}$-action on $\Gamma/\widetilde{\mathbb{O}}\oplus\mathbf{Z}_2^c$ is isomorphic to the $\mathbb{O}$-action on $\Gamma/\mathbb{O}\oplus\mathbf{Z}_2^c$. The groups $\mathbf{D}_4[\rho,\kappa_1]$, $\mathbb{T}[\tau,\rho_1]$ and $\mathbf{D}_3[\tau,\kappa]$ are all subgroups of $\mathbb{O}$. Let $\widetilde{\mathbf{D}_4}$ be the subgroup of $\widetilde{\mathbb{O}}$ isomorphic to $\mathbf{D}_4[\rho,\kappa_1]$, and similarly for $\widetilde{\mathbb{T}}$ and $\widetilde{\mathbf{D}_3}$. Then the $\widetilde{\mathbf{D}_4}$-action on $\Gamma/\widetilde{\mathbb{O}} \oplus \mathbf{Z}_2^c$ is isomorphic to the $\mathbf{D}_4[\rho,\kappa_1]$-action on $\Gamma/\mathbb{O} \oplus \mathbf{Z}_2^c$. Therefore, all fixed-point subspaces and symmetry results are identical, in particular, the projected skeletons in the two cases are identical. Similar comments hold for the groups $\widetilde{\mathbb{T}}$ and $\widetilde{\mathbf{D}_3}$.

Care must be taken when interpreting these result in physical space due to the different ways the tori are realized in $\mathbb{C}^4$. That is, the group orbit of FCC solutions is given by $\Gamma(x,x,x,x)$ whilst the group orbit of double-diamond solutions is given by $\Gamma(-x,x,x,x)$.

Corollary 4.4. *Let f be a $\mathbb{O} \oplus \mathbf{Z}_2^c \dotplus \mathbf{T}^3$-equivariant bifurcation problem on the FCC lattice. Let X_0 be the group orbit of double diamond solutions. Then there exists a small $\widetilde{\mathbf{D}_4}$- $\widetilde{\mathbb{T}}$- and $\widetilde{\mathbf{D}_3}$-equivariant perturbations, such that the perturbed flow on the perturbed group orbit of X_0 has homoclinic cycles.*

Proof. This follows from Corollary 2.8 and the discussion above. □

5 Body Centred Cubic Lattice

We consider the fundamental representation of $\Gamma = \mathbb{O} \oplus \mathbf{Z}_2 \dot{+} \mathbf{T}^3$ on the BCC lattice. In this case the representation of $\mathbb{O} \oplus \mathbf{Z}_2 \dot{+} \mathbf{T}^3$ is on $\mathbb{C}^6$ and is generated by:

$$\begin{aligned}
\rho_x(z) &= (z_4, z_1, z_2, z_3, , \overline{z_6}, z_5), \\
\rho_y(z) &= (z_5, \overline{z_4}, z_6, z_2, \overline{z_3}, z_1), \\
c(z) &= (\overline{z_1}, \overline{z_2}, \overline{z_3}, \overline{z_4}, \overline{z_5}, \overline{z_6}), \\
\theta(z) &= (e^{-\mathrm{i}(\theta_1+\theta_2+\theta_3)} z_1, e^{-\mathrm{i}(\theta_1+\theta_3)} z_2, e^{-\mathrm{i}(\theta_1-\theta_2)} z_3, e^{-\mathrm{i}\theta_1} z_4, e^{-\mathrm{i}(\theta_2+\theta_3)} z_5, e^{-\mathrm{i}\theta_2} z_6).
\end{aligned}$$

The notations is the same as the SC and FCC cases. The Γ-action on $\mathbb{C}^6$ has a large number of conjugacy classes of isotropy subgroups; there are sixteen with fixed-point subspace of dimension two or less [Callahan & Knobloch, 1997]. There are three conjugacy classes of translation free axial subgroups: $\mathbb{O} \oplus \mathbf{Z}_2^c$, $\widetilde{\mathbb{O}}$, and $\widetilde{\mathbf{D}}_4 \oplus \mathbf{Z}_2^c$ [Dionne et al., 1997; Callahan & Knobloch, 1997].

Callahan & Knobloch [1997] compute the generate form of a $\mathbb{O} \oplus \mathbf{Z}_2 \dot{+} \mathbf{T}^3$-equivariant vector field showing there is a quadratic equivariant. The presence of the quadratic term leads to a more involved analysis than the corresponding SC and FCC problems. Indeed, local to the origin, all solutions are unstable at bifurcation [Golubitsky et al., 1988]. Callahan & Knobloch [1997] adopt a well-known approach: they introduce an extra $\mathbf{Z}_2[-I]$ symmetry which removes all even order terms, then weakly unfold the quadratic term. The authors show the bifurcation problem with the extra $\mathbf{Z}_2[-I]$ symmetry has stable solutions with $\mathbb{O} \oplus \mathbf{Z}_2^c$ and $\widetilde{\mathbb{O}}$ symmetry, but not simultaneously. The solution with $\widetilde{\mathbf{D}}_4 \oplus \mathbf{Z}_2^c$ symmetry can never be stable. Upon unfolding the quadratic degeneracy the authors show the $\mathbb{O} \oplus \mathbf{Z}_2^c$ and $\widetilde{\mathbb{O}}$ symmetric solutions can be stable, but not simultaneously. We call the solution with $\mathbb{O} \oplus \mathbf{Z}_2^c$ symmetry the BCC solution. Our work concentrates on the BCC solution. As with the SC and FCC cases, this solution is normally hyperbolic and its group orbit is diffeomorphic to a 3-torus. We denote the group orbit by X_0.

We consider two perturbations of X_0, one with $\mathbb{T}[\tau, \rho_1]$ symmetry and the other with $\mathbf{D}_3[\tau, \kappa]$ symmetry. The notation is identical to that introduced previously. The action of $\mathbb{T}[\tau, \rho_1]$ on $\mathbb{C}^6$ is generated by

$$
\begin{aligned}
\tau z &= (z_4, \overline{z_5}, \overline{z_2}, \overline{z_6}, z_3, \overline{z_1}) \\
\rho_1 z &= (z_3, z_4, z_1, z_2, \overline{z_5}, \overline{z_6})
\end{aligned}
$$

and the action of $\mathbf{D}_3[\tau,\kappa]$ is generated by τ and

$$\kappa z = (z_6, z_2, z_5, \overline{z_4}, z_3, z_1).$$

By the Equivariant Persistence Theorem if the perturbation is sufficiently small X_0 persists to give a flow-invariant manifold X_ε that is equivariantly diffeomorphic to X_0.

There is an induced action of $\mathbf{D}_3[\tau,\kappa]$ and $\mathbb{T}[\tau,\rho_1]$ on X_0 and it is straightforward to verify that the generators of these groups act by:

$$\begin{aligned}
\tau(\theta_1,\theta_2,\theta_3) &= (\theta_2, \theta_1+\theta_2+\theta_3, -2\theta_2-\theta_3)\\
\kappa(\theta_1,\theta_2,\theta_3) &= (-\theta_1, -(\theta_1+\theta_2+\theta_3), 2\theta_1+\theta_3)\\
\rho_1(\theta_1,\theta_2,\theta_3) &= (\theta_1+\theta_3, -\theta_2, -\theta_3).
\end{aligned}$$

The induced actions of $\mathbf{D}_3[\tau,\kappa]$ and $\mathbb{T}[\tau,\rho_1]$ imply that:

$$\begin{aligned}
\mathscr{C}_{\mathbb{T}[\tau,\rho_1]} &= \{(0,0,0),(0,\pi,0),(\pi,0,0),(\pi,\pi,0),C_{(\theta,0,0)},C_{(0,\theta,0)},C_{(0,0,\theta)},\\
&\quad C_{(\theta,0,-\theta)},C_{(0,\theta,-\theta)},C_{(\theta,\theta,-\theta)},C_{(\theta,\pi,0)},C_{(\pi,\theta,0)},C_{(\theta,\theta,-2\theta)},\\
&\quad C_{(\theta+\pi,\theta,-2\theta)}\}.\\
\mathscr{C}_{\mathbf{D}_3[\tau,\kappa]} &= \{(0,0,0),(\pi,\pi,\pi),C_{(\theta,-\theta,0)},C_{(\theta,-\theta,\pi)},C_{(\theta,\theta,-\theta)},C_{(0,\theta,-2\theta)},\\
&\quad C_{(\pi,\theta,\pi-2\theta)},C_{(\theta,0,-2\theta)},C_{(\theta,\pi,\pi-2\theta)}\}.
\end{aligned}$$

The $\mathbf{D}_3[\tau,\kappa]$- and $\mathbb{T}[\tau,\rho_1]$-actions on X_0 induce actions on $\mathscr{C}_{\mathbf{D}_3[\tau,\kappa]}$ and $\mathscr{C}_{\mathbb{T}[\tau,\rho_1]}$

respectively. These actions are listed in Tables 10 and 11. In each case we list those elements which act nontrivially. In Table 11 we adopt the same notation to describe the elements of $\mathbb{T}[\tau,\rho_1]$ introduced in the FCC case. Using the group actions on X_0 and the entries of Tables 10 and 11 we compute the isotropy data for our perturbations. The isotropy data for the $\mathbf{D}_3[\tau,\kappa]$ group is contained in Table 12 and the isotropy data for the $\mathbb{T}[\tau,\rho_1]$ group is contained in Table 13.

The $\mathbf{D}_3[\tau,\kappa]$ case. The analysis now follows similar lines to the SC and FCC cases. Table 12 shows that

$$\mathrm{Stab}(C_{(\theta,\theta,-\theta)})/\mathrm{stab}(C_{(\theta,\theta,-\theta)}) \cong \mathbf{Z}_2$$

and it is trivial otherwise. Since

$$\mathrm{Stab}(C_{(\theta,\theta,-\theta)})/\mathrm{stab}(C_{(\theta,\theta,-\theta)})$$

acts by reflection on $C_{(\theta,\theta,-\theta)}$ there is an axis of reflection symmetry and this axis joins the two steady states: $(0,0,0)$ and (π,π,π), which are the two knots. Thus the connecting orbit $C_{(\theta,\theta,-\theta)}$ projects into the $\mathbf{D}_3[\tau,\kappa]$-orbit space as a arc joining the two knots. Since neither of the other connecting orbits have knots they project into the orbit space as topological circles joining the steady states that lie upon them. The complete connectivity relations are given by

1. $\overline{C}_{(\theta,\theta,-\theta)}$ connects $\overline{(0,0,0)}$ to $\overline{(\pi,\pi,\pi)}$,
2. $\overline{C}_{(\theta,-\theta,0)}$ connects $\overline{(0,0,0)}$ to itself
3. $\overline{C}_{(\theta,-\theta,\pi)}$ connects $\overline{(\pi,\pi,\pi)}$ itself.

Figure 3(a) illustrates the projected skeleton.

Theorem 5.1. *Let f be a $\mathbb{O} \oplus \mathbf{Z}_2^c \dot{+} \mathbf{T}^3$-equivariant bifurcation problem on the BCC lattice. Then there exists a $\mathbf{D}_3[\tau,\kappa]$-equivariant perturbation of f such that there exists two persistent steady states on the perturbed group orbit of the BCC solutions and homoclinic cycles to these steady states.*

Proof. This follows from Corollary 2.8. □

The $\mathbb{T}[\tau,\rho_1]$ case. The group action in Table 11 shows there are three representatives for the connecting orbits:

$$\overline{C}_{(\theta,0,0)}, \overline{C}_{(0,0,\theta)}, \overline{C}_{(\theta,\pi,0)},$$

The representatives for the steady states that lie on the connecting orbits are $\overline{(0,0,0)}$ and $\overline{(0,\pi,0)}$. The connection $\overline{C}_{(\theta,0,0)}$ has two knots given by the steady states $\overline{(0,0,0)}$ and $\overline{(0,\pi,0)}$. The connection $\overline{C}_{(\theta,\pi,0)}$ has two knots given by $\overline{(0,\pi,0)}$ and $\overline{(\pi,\pi,0)} = \overline{(0,\pi,0)}$. The connection $\overline{C}_{(0,0,\theta)}$ has no knots. The connectivity relations are:

1. $\overline{C}_{(\theta,0,0)}$ connects $\overline{(0,0,0)}$ to $\overline{(0,\pi,0)}$,
2. $\overline{C}_{(0,0,\theta)}$ connects $\overline{(0,0,0)}$ to itself,
3. $\overline{C}_{(\theta,\pi,0)}$ connects $\overline{(0,\pi,0)}$ to itself.

Although the connection $\overline{C}_{(\theta,\pi,0)}$ has two knots, they project into the orbit space as the same point. Figure 3(b) illustrates the projected skeleton.

Theorem 5.2. *Let f be a $\mathbb{O}\oplus\mathbf{Z}_2^c\dot{+}\mathbf{T}^3$-equivariant bifurcation problem on the BCC lattice. Then there exists a $\mathbb{T}[\tau,\rho_1]$-equivariant perturbation of f such that there exists two persistent steady states on the perturbed group orbit of the BCC solutions and homoclinic cycles to these steady states.*

Proof. This follows from Corollary 2.8. □

Although the projected skeletons $\mathbb{X}^p_{\mathbf{D}_3[\tau,\kappa]}$ and $\mathbb{X}^p_{\mathbb{T}[\tau,\rho_1]}$ are isomorphic, the dynamics resulting from these perturbations are not the same; the skeletons are not the same, with a different number of persistence steady states and connecting orbits.

5.1 24- and 48-dimensional representations

This section considers the high-dimensional representations of the SC, FCC and BCC lattices; that is, when the wavelength of the instabilities do not coincide with the periodicity of the lattice. The symmetry group of the lattices is unchanged: $\Gamma = \mathbb{O} \oplus \mathbf{Z}_2^c \dot{+} \mathbf{T}^3$. Dionne [1993] gives a complete classification of all Bravais lattices that support translation free absolutely irreducible representations of Γ. The cubic lattices support two types of twenty-four-dimensional representations, although only one type on the FCC lattice can be translation free, and each lattice supports a forty-eight-dimensional representation. The details of the wave vectors and defining conditions for these representations can be found in [Dionne, 1993; Dionne & Golubitsky, 1992].

Dionne [1993] provides a complete classification of the translation free axial planforms on the SC, FCC and BCC lattices. This information is summarized in Table 14. We indicate the generators for the twisted subgroups isomorphic to $\mathbb{O} \oplus \mathbf{Z}_2^c$ since our theory applies to these groups. On the SC lattice the translations are in the directions of the generators of the SC lattice, namely $\ell_1 = (1, 0, 0)$, $\ell_2 = (0, 1, 0)$ and $\ell_3 = (0, 0, 1)$. On the FCC and BCC lattices the situation is more subtle since the generators of these lattices are not so straightforward. The FCC lattice is generated by $\ell_1 = (1/2, 1/2, 0)$,

$\ell_2 = (-1/2, 1/2, 0)$ and $\ell_3 = (0, 1/2, 1/2)$ and the BCC lattice is generated by $\ell_1 = (1, 0, 0)$, $\ell_2 = (0, 1, 0)$ and $\ell_3 = (1/2, 1/2, 1/2)$. On these lattices the generators of the twisted subgroups are given in terms of the generators of the lattice. Complete details of these lattices can be found in [Dionne, 1993].

For certain axial solution in the 24- and 48-dimensional representations the results from the fundamental representations apply. Indeed, give any axial solution with $\Sigma = \mathbb{O} \oplus \mathbf{Z}_2^c$ symmetry the homogeneous space Γ/Σ is Γ-equivariantly diffeomorphic to the orbit space. Given any Lie subgroup $\Delta \subseteq \Gamma$ the abstract action of Δ on Γ/Σ is independence of the representation of Γ. This implies all symmetry-based results are identical to the fundamental representation and, in particular, the projected skeletons are the same. However, more is true as the following lemma shows:

Lemma 5.3. *Consider a* $\Gamma = \mathbb{O} \oplus \mathbf{Z}_2^c \dot{+} \mathbf{T}^3$*-equivariant bifurcation problem on a cubic lattice. Let* $\Sigma = \mathbb{O} \oplus \mathbf{Z}_2^c$ *and* $\widetilde{\Sigma} = \widetilde{\mathbb{O}} \oplus \mathbf{Z}_2^c$*, where* $\widetilde{\mathbb{O}}$ *is a twisted subgroup of* $\mathbb{O} \dot{+} \mathbf{T}^3$*. Then, the action of* $\mathbb{O}$ *on* Γ/Σ *is isomorphic to the action of* $\widetilde{\mathbb{O}}$ *on* $\Gamma/\widetilde{\Sigma}$*.*

Proof. Let the generators of $\mathbb{O}$ be ρ_x and ρ_y. Since $\widetilde{\mathbb{O}}$ is a twisted subgroups of $\mathbb{O} \dot{+} \mathbf{T}^3$ its generators have the form

$$(\rho_x, \alpha), \quad \text{and} \quad (\rho_y, \beta).$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ are elements of $\mathbf{T}^3$.

Since $\mathbb{O}$ and $\widetilde{\mathbb{O}}$ are isomorphic there exists $f : \mathbb{O} \to \widetilde{\mathbb{O}}$ such that

$$f(\gamma) = (\gamma, \phi),$$

for all $\gamma \in \mathbb{O}$. Here ϕ is the twist of the element γ in the group $\widetilde{\mathbb{O}}$. Define $A : \Gamma/\Sigma \to \Gamma/\widetilde{\Sigma}$ by

$$A\left((\gamma, \theta, \pm I)\Sigma\right) = (\gamma, \theta + \phi, \pm I)\widetilde{\Sigma}.$$

Here $\pm I$ represents the component of $\mathbf{Z}_2^c$. The map A is a linear isomorphism and we claim that:

$$A\left((\rho_x, 0, 0)(\gamma, \theta, \pm I)\Sigma\right) = (\rho_x, \alpha, 0)\, A\left((\gamma, \theta, \pm I)\Sigma\right). \tag{5}$$

Here we have regarded the elements of $\mathbb{O}$ as elements of $\mathbb{O} \oplus \mathbf{Z}_2^c \dot{+} \mathbf{T}^3$ with no translation and no inversion.

We begin with the left-hand side of Eq. (5)

$$\begin{aligned} A\left((\rho_x, 0, 0)(\gamma, \theta, \pm I)\Sigma\right) &= A\left((\rho_x\gamma, \rho_x\theta, \pm I)\Sigma\right) && (6) \\ &= (\rho_x\gamma, \rho_x\theta + \widehat{\phi}, \pm I)\widetilde{\Sigma} && (7) \end{aligned}$$

where $\widehat{\phi}$ is the twist of the element $\rho_x\gamma$ in $\widetilde{\mathbb{O}}$. (More precisely, it is the twist of $f(\rho_x\gamma)$ in $\widetilde{\mathbb{O}}$.)

Next consider the right-hand side of Eq. (5). A computation shows that

$$(\rho_x, \alpha, 0)\, A\left((\gamma, \theta, \pm I)\Sigma\right) = (\rho_x\gamma, \rho_x\theta + \rho_x\phi + \alpha, \pm I). \tag{8}$$

Now we compute

$$(\rho_x, \alpha, 0)(\gamma, \phi) = (\rho_x\gamma, \rho_x\phi + \alpha, 0).$$

So the twist of $f(\rho_x\gamma)$ is $\rho_x\phi + \alpha$ Using this computation in Eqs. (6) and (8) shows that

$$\begin{aligned}(\rho_x, \alpha, 0)\, A\left((\gamma, \theta, \pm I)\Sigma\right). &= (\rho_x\gamma, \rho_x\theta + \rho_x\phi + \alpha) \\ &= (\rho_x\gamma, \rho_x\theta + \widehat{\phi}).\end{aligned}$$

This verifies the claim. A similar computation shows that

$$A\left((\rho_y, 0, 0)(\gamma, \theta, \pm I)\Sigma\right) = (\rho_y, \beta, 0)\, A\left((\gamma, \theta, \pm I)\Sigma\right),$$

which proves the result. □

Using Lemma 5.3 we may apply the results for the fundamental representations of the cubic lattices to the 24- and 48-dimensional representations of the cubic lattices, provided the axial subgroups is isomorphic to $\mathbb{O} \oplus \mathbf{Z}_2^c$. Of course, the perturbations we choose have to be changed in the appropriate way. For example, consider the 24-dimensional type 1 representation of the SC lattice. There are two axial subgroups isomorphic to $\mathbb{O} \oplus \mathbf{Z}_2^c$, these are $\mathbb{O} \oplus \mathbf{Z}_2^c$ itself and $\widetilde{\mathbb{O}} \oplus \mathbf{Z}_2^c$. Let f be the isomorphism between $\mathbb{O}$ and $\widetilde{\mathbb{O}}$, then $\widetilde{\mathbb{O}}$ is generated by $f(\rho_x)$ and $f(\rho_y)$. Consider the subgroup $\mathbf{D}_3[\tau, \kappa]$ of $\mathbb{O}$. We proved in the fundamental representation of the SC lattice there exists $\mathbf{D}_3[\tau, \kappa]$-equivariant perturbations that give homoclinic cycles on the perturbed group orbit. We have already seen that these results generalize to the 24-dimensional type 1 representation when the axial subgroup is $\mathbb{O} \oplus \mathbf{Z}_2^c$. Consider the group $\widetilde{\mathbf{D}}_3[f(\tau), f(\kappa)]$ of $\widetilde{\mathbb{O}}$. By Lemma 5.3 the $\widetilde{\mathbf{D}}_3[f(\tau), f(\kappa)]$-action on $\Gamma/\widetilde{\mathbb{O}} \oplus \mathbf{Z}_2^c$ is isomorphic to the $\mathbf{D}_3[\tau, \kappa]$ action on $\Gamma/\mathbb{O} \oplus \mathbf{Z}_2^c$. This implies all symmetry based results are isomorphic and the projected skeletons are isomorphic. In particular, there exist $\widetilde{\mathbf{D}}_3[f(\tau), f(\kappa)]$-equivariant perturbations of a Γ-equivariant bifurcation problem in the 24-dimensional type 1 representation such that the solution with $\widetilde{\mathbb{O}} \oplus \mathbf{Z}_2^c$ symmetry has homoclinic cycles for perturbed flow.

We generalize these observations in the following theorem:

Theorem 5.4. *Let* $\Gamma = \mathbb{O} \oplus \mathbf{Z}_2^c \dotplus \mathbf{T}^3$, *let* $\Sigma = \mathbb{O} \oplus \mathbf{Z}_2^c$. *Let* $\widetilde{\Sigma}$ *be any* *twisted subgroup listed in Table 14 isomorphic to* Σ*. Let* ρ_x*,* ρ_y *denote a set of generators for* $\mathbb{O}$ *and let* (ρ_x, α) *and* (ρ_y, β) *denote a set of generators for* $\widetilde{\mathbb{O}} \subseteq \widetilde{\Sigma}$*. Let* f *be an isomorphism between* Σ *and* $\widetilde{\Sigma}$ *such that* $f(\rho_x) = (\rho_x, \alpha)$ *and* $f(\rho_y) = (\rho_y, \beta)$*.*

Then

1. *On the SC lattice there exist perturbations with* $\widetilde{\mathbf{D}_3}[f(\tau), f(\kappa)]$ *symmetry that give homoclinic cycles for the perturbed flow.*

2. *On the FCC lattice, homoclinic cycles exists for perturbations with* $\widetilde{\mathbf{D}_4}[f(\rho), f(\kappa_1)]$*,* $\widetilde{\mathbb{T}}[f(\tau), f(\rho_1)]$ *and* $\widetilde{\mathbf{D}}_3[f(\tau), f(\kappa)]$ *symmetry.*

3. *On the BCC lattice, homoclinic cycles exists for perturbations with* $\widetilde{\mathbb{T}}[f(\tau), f(\rho_1)]$ *and* $\widetilde{\mathbf{D}}_3[f(\tau), f(\kappa)]$ *symmetry.*

Proof. This follows from Lemma 5.3 and the results from the fundamental representations of Γ on the SC,FCC and BCC lattices. □

6 Conclusion

This paper presents examples of forced symmetry-breaking of SC, FCC and BCC planforms. The perturbations were chosen so the perturbed flow always had steady states and homoclinic cycles. Our examples are not limited to the fundamental representations of the cubic lattices and apply quite generally to all translation free irreducible representations of the cubic lattices.

For each perturbation we enumerated all projected skeletons, thus classifying all persistent steady states and heteroclinic orbits between the steady states that are forced to exist by the residual symmetry. Whilst this allows us to prove the existence of homoclinic cycles, the projected skeletons can support dynamics considerable more complex than this. Consider, for example, the $\mathbf{D}_4[\rho, \kappa_1]$- and $\mathbb{T}[\tau, \rho_1]$-equivariant perturbations on the FCC lattice. There exist scenarios where heteroclinic cycles can exist between the steady states on the projected skeletons in these cases. Indeed, Theorem 2.6 guarantees the existence of heteroclinic orbits between the steady states, but it does not guarantee these flows can be arranged to form a heteroclinic cycle. Indeed, although certain flows are permitted, they could be non-generic, Parker et al. [2006b] give an example on the hexagonal lattice where any flow on the projected skeleton resulting in a heteroclinic cycle must be non-hyperbolic. Further investigation of these elaborate dynamics requires detailed knowledge of the group invariants and equivariants. We have not examined the stability of any of our homoclinic cycles and again detailed knowledge of the group invariants and equivariants is required to answer the stability issues. Notwithstanding the restrictions on our results, they are still fairly general and provide a countable collection of homoclinic cycles on the cubic lattices. These results extend the general classification theory for perturbations of two-dimensional planforms presented in [Hou & Golubitsky, 1997; Parker et al., 2006a,b] to three dimensions. A more complete classification is underway [Parker, 2006].

Spatially periodic patterns arise in many different physical systems, with many examples being given in the introduction. There are three main areas where three-dimensional patterns are of interest: reaction-diffusion systems [De Wit et al., 1992; Gomes, 1999; Gunaratne et al., 1994; Callahan & Knobloch, 1999; Castets et al., 1990; Ouyang & Swinney, 1991; Ouyang et al., 1992; Turing, 1952; Zhou et al., 2002; Walgraef et al., 1982], the PA-MBO reaction [Orbán et al., 1999; Kurin-Csörgei et al., 1998; Münster et al., 1996; Münster, 1999; Steinbock et al., 1999] and certain nonlinear optical systems [D'Alessandro & Firth, 1992; Degtiarev & Vorontsov, 1996; Staliunas, 1998; Staliunas & J.Sánchez-Morcillo, 2000; Staliunas et al., 1997; Vorontsov & Samson, 1998; Vorontsov & Firth, 1994; Vorontsov & Karpov, 1997]. The literature on this subject is controversial and a definitive answer concerning the dimensionality of some patterns in these systems is still lacking, for example, the black-eye instabilities seen in the CIMA reaction [Gunaratne et al., 1994; Yang et al., 2002; Gomes, 1999], although such uncertainties are not limited to reaction-diffusion system and are present in both nonlinear optical systems and the PA-MBO reaction, see the literature cited above for details. Our results should shed some light onto the dimensionality of these patterns; we have shown that three-dimensional planforms on the cubic lattices are robust to small perturbations of the underlying symmetry-based modelling assumptions.

Our results are not limited to the systems mentioned above. Indeed they apply to any system exhibiting steady triply-periodic solutions.

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Tables

Table 1: Induced $\mathbf{D}_3[\tau,\kappa]$-action on $\mathscr{C}_{\mathbf{D}_3[\tau,\kappa]}$ for the SC lattice. Only those elements acting non-trivially are shown.

Element of $\mathscr{C}_{\mathbf{D}_3[\tau,\kappa]}$	Elements of $\mathbf{D}_3[\tau,\kappa]$	Action
$C_{(\theta,-\theta,0)}$	$\tau,\ \kappa$	$C_{(0,\theta,\theta)}$
	$\tau\kappa,\ \tau^2$	$C_{(\theta,0,\theta)}$
$C_{(\theta,-\theta,\pi)}$	$\tau,\ \kappa$	$C_{(\pi,\theta,\theta)}$
	$\tau\kappa,\ \tau^2$	$C_{(\theta,\pi,\theta)}$
$C_{(\theta,0,\theta)}$	$\tau,\ \tau\kappa$	$C_{(\theta,-\theta,0)}$
	$\tau^2,\ \tau^2\kappa$	$C_{(0,\theta,\theta)}$
$C_{(\theta,\pi,\theta)}$	$\tau,\ \tau\kappa$	$C_{(\theta,-\theta,\pi)}$
	$\tau^2,\ \tau^2\kappa$	$C_{(\pi,\theta,\theta)}$
$C_{(0,\theta,\theta)}$	$\tau,\ \tau^2\kappa$	$C_{(\theta,0,\theta)}$
	$\tau^2,\ \kappa$	$C_{(\theta,-\theta,0)}$
$C_{(\pi,\theta,\theta)}$	$\tau,\ \kappa$	$C_{(\theta,\pi,\theta)}$
	$\tau^2,\ \tau^2\kappa$	$C_{(\theta,-\theta,\pi)}$

Table 2: Isotropy data for $C \in \mathscr{C}_{\mathbf{D}_3[\tau,\kappa]}$ on the SC lattice.

$C \in \mathscr{C}_{\mathbf{D}_3[\tau,\kappa]}$	$\mathrm{stab}(C)$	$\mathrm{Stab}(C)$
$(0,0,0)$	$\mathbf{D}_3[\tau,\kappa]$	$\mathbf{D}_3[\tau,\kappa]$
(π,π,π)	$\mathbf{D}_3[\tau,\kappa]$	$\mathbf{D}_3[\tau,\kappa]$
$C_{(\theta,-\theta,0)}$	$\mathbf{Z}_2[\tau^2\kappa]$	$\mathbf{Z}_2[\tau^2\kappa]$
$C_{(\theta,-\theta,\pi)}$	$\mathbf{Z}_2[\tau^2\kappa]$	$\mathbf{Z}_2[\tau^2\kappa]$
$C_{(\theta,0,-\theta)}$	$\mathbf{Z}_2[\kappa]$	$\mathbf{Z}_2[\kappa]$
$C_{(\theta,\pi,-\theta)}$	$\mathbf{Z}_2[\kappa]$	$\mathbf{Z}_2[\kappa]$
$C_{(0,\theta,-\theta)}$	$\mathbf{Z}_2[\tau\kappa]$	$\mathbf{Z}_2[\tau\kappa]$
$C_{(\pi,\theta,-\theta)}$	$\mathbf{Z}_2[\tau\kappa]$	$\mathbf{Z}_2[\tau\kappa]$
$C_{(\theta,\theta,-\theta)}$	$\mathbf{Z}_3[\tau]$	$\mathbf{D}_3[\tau,\kappa]$

Table 3: Induced $\mathbf{D}_4[\rho, \kappa_1]$-action on $\mathscr{C}_{\mathbf{D}_4[\rho,\kappa_1]}$ for the FCC lattice. Only those elements acting nontrivially are shown.

Element of $\mathscr{C}_{\mathbf{D}_4[\rho,\kappa_1]}$	Elements of $\mathbf{D}_4[\rho, \kappa_1]$	Action
$(0, 0, \pi)$	$\rho,\ \rho^3,\ \rho\kappa_1,\ \rho^3\kappa_1$	(π, π, π)
(π, π, π)	$\rho,\ \rho^3,\ \rho\kappa_1,\ \rho^3\kappa_1$	$(0, 0, \pi)$
$(3\pi/2, \pi/2, \pi)$	$\rho, \rho^3, \kappa_1, \rho^2\kappa_1$	$(\pi/2, 3\pi/2, \pi)$
$(\pi/2, 3\pi/2, \pi)$	$\rho, \rho^3, \kappa_1, \rho^2\kappa_1$	$(3\pi/2, \pi/2, \pi)$
$C_{(0,0,\theta)}$	$\rho,\ \rho^3,\ \rho\kappa_1,\ \rho^3\kappa_1$	$C_{(\theta,\theta,-\theta)}$
$C_{(\pi,\pi,\theta)}$	$\rho,\ \rho^3,\ \rho\kappa_1,\ \rho^3\kappa_1$	$C_{(\theta,\theta,\pi-\theta)}$
$C_{(\theta,\theta,0)}$	$\rho,\ \rho^3,\ \kappa_1,\ \rho^2\kappa_1,$	$C_{(\theta,\theta,-2\theta)}$
$C_{(\theta,\theta,-\theta)}$	$\rho,\ \rho^3,\ \rho\kappa_1,\ \rho^3\kappa_1$	$C_{(0,0,\theta)}$
$C_{(\theta,\theta,-2\theta)}$	$\rho,\ \rho^3,\ \kappa_1,\ \rho^2\kappa_1,$	$C_{(\theta,\theta,0)}$
$C_{(\theta,\theta,\pi-\theta)}$	$\rho,\ \rho^3,\ \rho\kappa_1,\ \rho^3\kappa_1$	$C_{(\pi,\pi,\theta)}$
$C_{(\theta+\pi,\theta,\pi)}$	$\rho,\ \rho^3,\ \kappa_1,\ \rho^2\kappa_1,$	$C_{(\theta+\pi,\theta,-2\theta)}$
$C_{(\theta+\pi,\theta,-2\theta)}$	$\rho,\ \rho^3,\ \kappa_1,\ \rho^2\kappa_1,$	$C_{(\theta+\pi,\theta,\pi)}$

Table 4: Induced $\mathbb{T}[\tau,\rho_1]$-action on $\mathscr{C}_{\mathbb{T}[\tau,\rho_1]}$ for the FCC lattice. Only those elements acting nontrivially are shown.

Element of $\mathscr{C}_{\mathbb{T}[\tau,\rho_1]}$	Elements of $\mathbb{T}[\tau,\rho_1]$	Action
$(3\pi/2,\pi/2,\pi)$	$\tau,\tau^2,\tau_1,\tau_1^2,\tau_2,\tau_2^2,\tau_3,\tau_3^2$	$(\pi/2,3\pi/2,\pi)$
$(\pi/2,3\pi/2,\pi)$	$\tau,\tau^2,\tau_1,\tau_1^2,\tau_2,\tau_2^2,\tau_3,\tau_3^2$	$(3\pi/2,\pi/2,\pi)$
$C_{(\theta,\theta,0)}$	$\tau_1^2,\ \tau_2,\ \tau^2,\ \tau_3$	$C_{(\theta,\theta,-2\theta)}$
	$\tau_1,\ \tau_2^2,\ \tau,\ \tau_3^2$	$C_{(\theta,-\theta,0)}$
$C_{(\theta,-\theta,0)}$	$\tau_1,\ \tau_2^2,\ \tau,\ \tau_3^2$	$C_{(\theta,\theta,-2\theta)}$
	$\tau_1^2,\ \tau_2,\ \tau^2,\ \tau_3$	$C_{(\theta,\theta,0)}$
$C_{(\theta,-\theta,\pi)}$	$\tau_1,\ \tau_2^2,\ \tau,\ \tau_3^2$	$C_{(\theta+\pi,\theta,-2\theta)}$
	$\tau_1^2,\ \tau_2,\ \tau^2,\ \tau_3$	$C_{(\theta+\pi,\theta,\pi)}$
$C_{(\theta,-\theta,2\theta)}$	$\rho_2\ ,\ \tau_2^2,\ \tau$	$C_{(\theta,3\theta,-2\theta)}$
	$\rho_1,\ \tau,\ \tau_3$	$C_{(\theta,-\theta,-2\theta)}$
	$\rho_3,\ \tau_2,\ \tau_3^2$	$C_{(3\theta,\theta,-2\theta)}$
$C_{(\theta,-\theta,-2\theta)}$	$\rho_2,\ \tau_1,\ \tau_3$	$C_{(3\theta,\theta,-2\theta)}$
	$\rho_1,\ \tau^2,\ \tau_3^2$	$C_{(\theta,-\theta,2\theta)}$
	$\rho_3,\ \tau_1^2,\ \tau$	$C_{(\theta,3\theta,-2\theta)}$
$C_{(3\theta,\theta,-2\theta)}$	$\rho_3,\ \tau_2^2,\ \tau_3$	$C_{(\theta,-\theta,2\theta)}$
	$\rho_1,\ \tau_1,\ \tau_2$	$C_{(\theta,3\theta,-2\theta)}$
	$\rho_2,\ \tau_1^2,\ \tau_3^2$	$C_{(\theta,-\theta,-2\theta)}$
$C_{(\theta,3\theta,-2\theta)}$	$\rho_3,\ \tau_1,\ \tau^2$	$C_{(\theta,-\theta,-2\theta)}$
	$\rho_1,\ \tau_1^2,\ \tau_2^2$	$C_{(3\theta,\theta,-2\theta)}$
	$\rho_2,\ \tau_2,\ \tau$	$C_{(\theta,-\theta,2\theta)}$
$C_{(\theta,\theta,-2\theta)}$	$\tau_1,\ \tau_2^2,\ \tau,\ \tau_3^2$	$C_{(\theta,\theta,0)}$
	$\tau_1^2,\ \tau_2,\ \tau^2,\ \tau_3$	$C_{(\theta,-\theta,0)}$
$C_{(\theta+\pi,\theta,\pi)}$	$\tau_1,\ \tau_2^2,\ \tau,\ \tau_3^2$	$C_{(\theta+\pi,\theta,-2\theta)}$
	$\tau_1,\ \tau_2^2,\ \tau,\ \tau_3^2$	$C_{(\theta,-\theta,\pi)}$
$C_{(\theta+\pi,\theta,-2\theta)}$	$\tau_1,\ \tau_2^2,\ \tau,\ \tau_3^2$	$C_{(\theta+\pi,\theta,\pi)}$
	$\tau_1,\ \tau_2^2,\ \tau,\ \tau_3^2$	$C_{(\theta,-\theta,\pi)}$

Table 5: Induced $\mathbf{D}_3[\tau,\kappa]$-action on $\mathscr{C}_{\mathbf{D}_3[\tau,\kappa]}$ for the FCC lattice. Only those elements acting nontrivially are shown.

Element of $\mathscr{C}_{\mathbf{D}_3[\tau,\kappa]}$	Elements of $\mathbf{D}_3[\tau,\kappa]$	Action
$C_{(0,\theta,0)}$	$\tau,\ \tau\kappa$	$C_{(0,\theta,-\theta)}$
	$\tau^2,\ \kappa$	$C_{(0,0,\theta)}$
$C_{(\pi,\theta,0)}$	$\tau,\ \tau\kappa$	$C_{(\pi,\theta,\pi-\theta)}$
	$\tau^2,\ \kappa$	$C_{(\pi,\pi,\theta)}$
$C_{(0,\theta,-\theta)}$	$\tau,\ \tau^2\kappa$	$C_{(0,0,\theta)}$
	$\tau^2,\ \tau\kappa$	$C_{(0,\theta,0)}$
$C_{(\pi,\theta,\pi-\theta)}$	$\tau,\ \tau^2\kappa$	$C_{(\pi,\pi,\theta)}$
	$\tau^2,\ \tau\kappa$	$C_{(\pi,\theta,0)}$
$C_{(0,0,\theta)}$	$\tau,\ \kappa$	$C_{(0,\theta,0)}$
	$\tau^2,\ \tau^2\kappa$	$C_{(0,\theta,-\theta)}$
$C_{(\pi,\pi,\theta)}$	$\tau,\ \kappa$	$C_{(\pi,\theta,0)}$
	$\tau^2,\ \tau^2\kappa$	$C_{(\pi,\theta,\pi-\theta)}$

Table 6: Isotropy data for $C \in \mathscr{C}_{\mathbf{D}_4[\rho,\kappa_1]}$ on the FCC lattice.

$C \in \mathscr{C}_{\mathbf{D}_4[\rho,\kappa_1]}$	stab(C)	Stab(C)
$(0,0,0)$	$\mathbf{D}_4[\rho,\kappa_1]$	$\mathbf{D}_4[\rho,\kappa_1]$
$(0,0,\pi)$	$\mathbf{D}_2[\rho^2,\kappa_1\rho^3]$	$\mathbf{D}_2[\rho^2,\kappa_1\rho^3]$
$(\pi,\pi,0)$	$\mathbf{D}_4[\rho,\kappa_1]$	$\mathbf{D}_4[\rho,\kappa_1]$
(π,π,π)	$\mathbf{D}_2[\rho^2,\kappa_1]$	$\mathbf{D}_2[\rho^2,\kappa_1\rho^3]$
$(3\pi/2,\pi/2,\pi)$	$\mathbf{D}_2[\rho^2,\kappa_1\rho^3]$	$\mathbf{D}_2[\rho^2,\kappa_1\rho^3]$
$(\pi/2,3\pi/2,\pi)$	$\mathbf{D}_2[\rho^2,\kappa_1\rho^3]$	$\mathbf{D}_2[\rho^2,\kappa_1\rho^3]$
$C_{(0,0,\theta)}$	$\mathbf{Z}_2[\kappa_1\rho^2]$	$\mathbf{D}_2[\rho^2,\kappa\rho^2]$
$C_{(\pi,\pi,\theta)}$	$\mathbf{Z}_2[\kappa_1\rho^2]$	$\mathbf{D}_2[\rho^2,\kappa\rho^2]$
$C_{(\theta,\theta,0)}$	$\mathbf{Z}_2[\kappa_1\rho^3]$	$\mathbf{D}_2[\rho^2,\kappa_1\rho^3]$
$C_{(\theta,-\theta,0)}$	$\mathbf{Z}_4[\rho]$	$\mathbf{D}_4[\rho,\kappa_1]$
$C_{(\theta,-\theta,\pi)}$	$\mathbf{Z}_2[\rho^2]$	$\mathbf{D}_4[\rho,\kappa_1]$
$C_{(\theta,\theta,-\theta)}$	$\mathbf{Z}_2[\kappa_1]$	$\mathbf{D}_2[\rho^2,\kappa_1]$
$C_{(\theta,\theta,-2\theta)}$	$\mathbf{Z}_2[\kappa_1\rho]$	$\mathbf{D}_2[\rho^2,\kappa_1\rho^3]$
$C_{(\theta,\theta,\pi-\theta)}$	$\mathbf{Z}_2[\kappa_1]$	$\mathbf{D}_2[\rho^2,\kappa_1]$
$C_{(\theta+\pi,\theta,\pi)}$	$\mathbf{Z}_2[\kappa_1\rho^3]$	$\mathbf{D}_2[\rho^2,\kappa_1\rho^3]$
$C_{(\theta+\pi,\theta,-2\theta)}$	$\mathbf{Z}_2[\kappa_1\rho^3]$	$\mathbf{D}_2[\rho^2,\kappa_1\rho^3]$

Table 7: Isotropy data for $C \in \mathscr{C}_{\mathbb{T}[\tau,\rho_1]}$ on the FCC lattice.

$C \in \mathscr{C}_{\mathbb{T}[\tau,\rho_1]}$	stab(C)	Stab(C)
$(0,0,0)$	$\mathbb{T}[\tau,\rho_1]$	$\mathbb{T}[\tau,\rho_1]$
$(\pi,\pi,0)$	$\mathbb{T}[\tau,\rho_1]$	$\mathbb{T}[\tau,\rho_1]$
$(3\pi/2,\pi/2,\pi)$	$\mathbf{D}_2[\rho_1,\rho_2]$	$\mathbf{D}_2[\rho_1,\rho_2]$
$(\pi/2,3\pi/2,\pi)$	$\mathbf{D}_2[\rho_1,\rho_2]$	$\mathbf{D}_2[\rho_1,\rho_2]$
$C_{(\theta,\theta,0)}$	$\mathbf{Z}_2[\rho_1]$	$\mathbf{D}_2[\rho_1,\rho_2]$
$C_{(\theta,-\theta,0)}$	$\mathbf{Z}_2[\rho_1]$	$\mathbf{D}_2[\rho_1,\rho_2]$
$C_{(\theta,-\theta,\pi)}$	$\mathbf{Z}_2[\rho_1]$	$\mathbf{D}_2[\rho_1,\rho_2]$
$C_{(\theta,-\theta,2\theta)}$	$\mathbf{Z}_3[\tau_1]$	$\mathbf{Z}_3[\tau_1]$
$C_{(\theta,-\theta,-2\theta)}$	$\mathbf{Z}_3[\tau_2]$	$\mathbf{Z}_3[\tau_2]$
$C_{(3\theta,\theta,-2\theta)}$	$\mathbf{Z}_3[\tau_3]$	$\mathbf{Z}_3[\tau_3]$
$C_{(\theta,3\theta,-2\theta)}$	$\mathbf{Z}_3[\tau_3]$	$\mathbf{Z}_3[\tau_3]$
$C_{(\theta,\theta,-2\theta)}$	$\mathbf{Z}_2[\rho_3]$	$\mathbf{D}_2[\rho_1,\rho_2]$
$C_{(\theta+\pi,\theta,\pi)}$	$\mathbf{Z}_2[\rho_2]$	$\mathbf{D}_2[\rho_1,\rho_2]$
$C_{(\theta+\pi,\theta,-2\theta)}$	$\mathbf{Z}_2[\rho_3]$	$\mathbf{D}_2[\rho_1,\rho_2]$

Table 8: Isotropy data for $C \in \mathscr{C}_{\mathbf{D}_3[\tau,\kappa]}$ on the FCC lattice.

$C \in \mathscr{C}_{\mathbf{D}_3[\tau,\kappa]}$	$\mathrm{stab}(C)$	$\mathrm{Stab}(C)$
$(0,0,0)$	$\mathbf{D}_3[\tau,\kappa]$	$\mathbf{D}_3[\tau,\kappa]$
$(\pi,\pi,0)$	$\mathbf{D}_3[\tau,\kappa]$	$\mathbf{D}_3[\tau,\kappa]$
$C_{(0,\theta,0)}$	$\mathbf{Z}_2[\tau^2\kappa]$	$\mathbf{Z}_2[\tau^2\kappa]$
$C_{(\pi,\theta,0)}$	$\mathbf{Z}_2[\tau^2\kappa]$	$\mathbf{Z}_2[\tau^2\kappa]$
$C_{(0,\theta,-\theta)}$	$\mathbf{Z}_2[\kappa]$	$\mathbf{Z}_2[\kappa]$
$C_{(\pi,\theta,\pi-\theta)}$	$\mathbf{Z}_2[\kappa]$	$\mathbf{Z}_2[\kappa]$
$C_{(0,0,\theta)}$	$\mathbf{Z}_2[\tau\kappa]$	$\mathbf{Z}_2[\tau\kappa]$
$C_{(\pi,\pi,\theta)}$	$\mathbf{Z}_2[\tau\kappa]$	$\mathbf{Z}_2[\tau\kappa]$
$C_{(3\theta,\theta,-2\theta)}$	$\mathbf{Z}_2[\tau\kappa]$	$\mathbf{D}_3[\tau,\kappa]$

Table 9: Knots on $\mathbf{D}[\rho, \kappa_1]$-orbit representatives for the connecting orbits for the FCC lattice.

Element of $\mathscr{C}_{\mathbf{D}_4[\rho,\kappa_1]}$	Knots
$\overline{C}_{(0,0,\theta)}$	$\overline{(0,0,0)}$, $\overline{(0,0,\pi)}$
$\overline{C}_{(\pi,\pi,\theta)}$	$\overline{(\pi,\pi,0)}$, $\overline{(0,0,\pi)}$
$\overline{C}_{(\theta,\theta,0)}$	$\overline{(0,0,0)}$, $\overline{(\pi,\pi,0)}$
$\overline{C}_{(\theta,-\theta,0)}$	$\overline{(0,0,0)}$, $\overline{(\pi,\pi,0)}$
$\overline{C}_{(\theta,-\theta,\pi)}$	$\overline{(0,0,\pi)}$, $\overline{(3\pi/2,\pi/2,\pi)}$
$\overline{C}_{(\theta+\pi,\theta,\pi)}$	$\overline{(3\pi/2,\pi/2,\pi)}$,$\overline{(3\pi/2,\pi/2,\pi)}$

Table 10: Induced $\mathbf{D}_3[\tau,\kappa]$-action on $\mathscr{C}_{\mathbf{D}_3[\tau,\kappa]}$ on the BCC lattice. Only those elements acting nontrivially are shown.

Element of $\mathscr{C}_{\mathbf{D}_3[\tau,\kappa]}$	Elements of $\mathbf{D}_3[\tau,\kappa]$	Action
$C_{(\theta,-\theta,0)}$	$\tau\kappa,\ \tau^2,$	$C_{(0,\theta,-2\theta)}$
	$\kappa,\ \tau$	$C_{(\theta,0,-2\theta)}$
$C_{(\theta,-\theta,\pi)}$	$\tau\kappa,\ \tau^2,$	$C_{(\pi,\theta,\pi-2\theta)}$
	$\kappa,\ \tau$	$C_{(\theta,\pi,\pi-2\theta)}$
$C_{(0,\theta,-2\theta)}$	$\tau\kappa,\ \tau,$	$C_{(\theta,-\theta,0)}$
	$\tau^2\kappa,\ \tau^2$	$C_{(\theta,0,-2\theta)}$
$C_{(\pi,\theta,\pi-2\theta)}$	$\tau\kappa,\ \tau,$	$C_{(\theta,-\theta,\pi)}$
	$\tau^2\kappa,\ \tau^2$	$C_{(\theta+\pi,\theta,-2\theta)}$
$C_{(\theta,0,-2\theta)}$	$\kappa,\ \tau^2$	$C_{(\theta,-\theta,0)}$
	$\tau^2\kappa,\ \tau$	$C_{(0,\theta,-2\theta)}$
$C_{(\theta,\pi,\pi-2\theta)}$	$\kappa,\ \tau^2$	$C_{(\theta,-\theta,\pi)}$
	$\tau^2\kappa,\ \tau$	$C_{(\pi,\theta,\pi-2\theta)}$

Table 11: Induced $\mathbb{T}[\tau,\rho_1]$-action on $\mathscr{C}_{\mathbb{T}[\tau,\rho_1]}$ for the BCC lattice. Only those elements acting nontrivially are shown.

Element of $\mathscr{C}_{\mathbb{T}[\tau,\rho_1]}$	Elements of $\mathbb{T}[\tau,\rho_1]$	Action
$(0,\pi,0)$	$\tau_1,\ \tau_2^2,\ \tau,\ \tau_3^2$	$(\pi,\pi,0)$
	$\tau_1^2,\ \tau_2,\ \tau^2,\ \tau_3$	$(\pi,0,0)$
$(\pi,0,0)$	$\tau_1^2,\ \tau_2,\ \tau^2,\ \tau_3$	$(\pi,\pi,0)$
	$\tau_1,\ \tau_2^2,\ \tau,\ \tau_3^2$	$(0,\pi,0)$
$(\pi,\pi,0)$	$\tau_1^2,\ \tau_2,\ \tau^2,\ \tau_3$	$(0,\pi,0)$
	$\tau_1,\ \tau_2^2,\ \tau,\ \tau_3^2$	$(\pi,0,0)$
$C_{(\theta,0,0)}$	$\tau_1^2,\ \tau_2,\ \tau^2,\ \tau_3$	$C_{(\theta,\theta,-2\theta)}$
	$\tau_1,\ \tau_2^2,\ \tau,\ \tau_3^2$	$C_{(0,\theta,0)}$
$C_{(0,\theta,0)}$	$\tau_1,\ \tau_2^2,\ \tau,\ \tau_3^2$	$C_{(\theta,\theta,-2\theta)}$
	$\tau_1^2,\ \tau_2,\ \tau^2,\ \tau_3$	$C_{(\theta,0,0)}$
$C_{(0,0,\theta)}$	$\rho_2,\tau_2,\ \tau$	$C_{(0,\theta,-\theta)}$
	$\rho_1,\tau^2,\ \tau_3^2$	$C_{(\theta,0,-\theta)}$
	$\rho_3,\ \tau_2^2,\ \tau_3$	$C_{(\theta,\theta,-\theta)}$
$C_{(\theta,0,-\theta)}$	$\rho_2,\ \tau_1^2,\ \tau_3^2$	$C_{(\theta,\theta,-\theta)}$
	$\rho_1,\ \tau,\ \tau_3$	$C_{(0,0,\theta)}$
	$\rho_3,\ \tau_1,\ \tau^2$	$C_{(0,\theta,-\theta)}$
$C_{(0,\theta,-\theta)}$	$\rho_3,\ \tau_1^2,\ \tau$	$C_{(\theta,0,-\theta)}$
	$\rho_1,\ \tau_1,\ \tau_2$	$C_{(\theta,\theta,-\theta)}$
	$\rho_2,\ \tau_2^2,\ \tau^2$	$C_{(0,0,\theta)}$
$C_{(\theta,\theta,-\theta)}$	$\rho_3,\ \tau_2,\ \tau_3^2$	$C_{(0,0,\theta)}$
	$\rho_1,\ \tau_1^2,\ \tau_2^2$	$C_{(0,\theta,-\theta)}$
	$\rho_2,\ \tau_1,\ \tau_3$	$C_{(\theta,0,-\theta)}$
$C_{(\theta,\pi,0)}$	$\tau_1^2,\ \tau_2,\ \tau^2,\ \tau_3$	$C_{(\theta+\pi,\theta,-2\theta)}$
	$\tau_1,\ \tau_2^2,\ \tau,\ \tau_3^2$	$C_{(\pi,\theta,0)}$
$C_{(\pi,\theta,0)}$	$\tau_1,\ \tau_2^2,\ \tau,\ \tau_3^2$	$C_{(\theta+\pi,\theta,-2\theta)}$
	$\tau_1^2,\ \tau_2,\ \tau^2,\ \tau_3$	$C_{(\theta,\pi,0)}$
$C_{(\theta,\theta,-2\theta)}$	$\tau_1^2,\ \tau_2,\ \tau^2,\ \tau_3$	$C_{(0,\theta,0)}$
	$\tau_1,\ \tau_2^2,\ \tau,\ \tau_3^2$	$C_{(\theta,0,0)}$
$C_{(\theta+\pi,\theta,-2\theta)}$	$\tau_1^2,\ \tau_2,\ \tau^2,\ \tau_3$	$C_{(\pi,\theta,0)}$
	$\tau_1,\ \tau_2^2,\ \tau,\ \tau_3^2$	$C_{(\theta,\pi,0)}$

Table 12: Isotropy data for $C \in \mathscr{C}_{\mathbf{D}_3[\tau,\kappa]}$ for the BCC lattice.

$C \in \mathscr{C}_{\mathbf{D}_3}$	stab(C)	
$(0,0,0)$	$\mathbf{D}_3[\tau,\kappa]$	$\mathbf{D}_3[\tau,\kappa]$
(π,π,π)	$\mathbf{D}_3[\tau,\kappa]$	$\mathbf{D}_3[\tau,\kappa]$
$C_{(\theta,-\theta,0)}$	$\mathbf{Z}_2[\tau^2\kappa]$	$\mathbf{Z}_2[\tau^2\kappa]$
$C_{(\theta,-\theta,\pi)}$	$\mathbf{Z}_2[\tau^2\kappa]$	$\mathbf{Z}_2[\tau^2\kappa]$
$C_{(\theta,\theta,-\theta)}$	$\mathbf{Z}_3[\tau]$	$\mathbf{D}_3[\tau,\kappa]$
$C_{(0,\theta,-2\theta)}$	$\mathbf{Z}_2[\kappa]$	$\mathbf{Z}_2[\kappa]$
$C_{(\pi,\theta,\pi-2\theta)}$	$\mathbf{Z}_2[\kappa]$	$\mathbf{Z}_2[\kappa]$
$C_{(\theta,0,-2\theta)}$	$\mathbf{Z}_2[\tau\kappa]$	$\mathbf{Z}_2[\tau\kappa]$
$C_{(\theta,\pi,\pi-2\theta)}$	$\mathbf{Z}_2[\tau\kappa]$	$\mathbf{Z}_2[\tau\kappa]$

Table 13: Isotropy data for $C \in \mathscr{C}_{\mathbb{T}[\tau,\rho_1]}$ for the BCC lattice.

$C \in \mathscr{C}_{\mathbb{T}[\tau,\rho_1]}$	stab(C)	Stab(C)
$(0,0,0)$	$\mathbb{T}[\tau,\kappa_1]$	$\mathbb{T}[\tau,\kappa_1]$
$(0,\pi,0)$	$\mathbf{D}_2[\rho_1,\rho_2]$	$\mathbf{D}_2[\rho_1,\rho_2]$
$(\pi,0,0)$	$\mathbf{D}_2[\rho_1,\rho_2]$	$\mathbf{D}_2[\rho_1,\rho_2]$
$(\pi,\pi,0)$	$\mathbf{D}_2[\rho_1,\rho_2]$	$\mathbf{D}_2[\rho_1,\rho_2]$
$C_{(\theta,0,0)}$	$\mathbf{Z}_2[\rho_1]$	$\mathbf{D}_2[\rho_1,\rho_2]$
$C_{(0,\theta,0)}$	$\mathbf{Z}_2[\rho_2]$	$\mathbf{D}_2[\rho_1,\rho_2]$
$C_{(0,0,\theta)}$	$\mathbf{Z}_3[\tau_1]$	$\mathbf{Z}_3[\tau_1]$
$C_{(\theta,0,-\theta)}$	$\mathbf{Z}_3[\tau_2]$	$\mathbf{Z}_3[\tau_2]$
$C_{(0,\theta,-\theta)}$	$\mathbf{Z}_3[\tau_3]$	$\mathbf{Z}_3[\tau_3]$
$C_{(\theta,\theta,-\theta)}$	$\mathbf{Z}_3[\tau]$	$\mathbf{Z}_3[\tau]$
$C_{(\theta,\pi,0)}$	$\mathbf{Z}_2[\rho_1]$	$\mathbf{D}_2[\rho_1,\rho_2]$
$C_{(\pi,\theta,0)}$	$\mathbf{Z}_2[\rho_2]$	$\mathbf{D}_2[\rho_1,\rho_2]$
$C_{(\theta,\theta,-2\theta)}$	$\mathbf{Z}_2[\rho_3]$	$\mathbf{D}_2[\rho_1,\rho_2]$
$C_{(\theta+\pi,\theta,-2\theta)}$	$\mathbf{Z}_2[\rho_3]$	$\mathbf{D}_2[\rho_1,\rho_2]$

Table 14: Translation free axial subgroups of the 24- and 48-dimensional representations of the cubic lattices. Generators are only shown for groups isomorphic to $\mathbb{O}$ which has generators ρ_x and ρ_y. Superscripts are used to indicated groups with different generators.

Lattice	Dimension	Axial Subgroup
SC	24 Type 1	$\mathbb{O} \oplus \mathbf{Z}_2^c$
		$\widetilde{\mathbb{O}}[(\rho_x,(\frac{1}{2},\frac{1}{2},0)),(\rho_y,(0,\frac{1}{2},\frac{1}{2}))] \oplus \mathbf{Z}_2^c$
		$\widetilde{\mathbb{O}}[(\rho_x,(\frac{1}{4},\frac{-1}{2},0)),(\rho_y,(0,\frac{1}{4},0))]$
		$\widetilde{\mathbb{T}} \oplus \mathbf{Z}_2^c$
		$\widetilde{\mathbf{D}}_3 \oplus \mathbf{Z}_2^c$
	24 Type 2	$\mathbb{O} \oplus \mathbf{Z}_2^c$
		$\widetilde{\mathbb{O}}[(\rho_x,(\frac{1}{2},0,0)),(\rho_y,(0,\frac{1}{2},0))] \oplus \mathbf{Z}_2^c$
		$\widetilde{\mathbb{O}}[(\rho_x,(\frac{1}{4},\frac{-1}{2},0)),(\rho_y,(0,\frac{1}{4},0))]$
		$\widetilde{\mathbb{T}}^a \oplus \mathbf{Z}_2^c$
	48	$\mathbb{O} \oplus \mathbf{Z}_2^c$
		$\widetilde{\mathbb{O}}[(\rho_x,(\frac{1}{2},\frac{1}{2},0)),(\rho_y,(0,\frac{1}{2},\frac{1}{2}))] \oplus \mathbf{Z}_2^c$
		$\widetilde{\mathbb{O}}[(\rho_x,(\frac{1}{2},0,0)),(\rho_y,(0,\frac{1}{2},0))] \oplus \mathbf{Z}_2^c$
		$\widetilde{\mathbb{O}}[(\rho_x,(0,\frac{1}{2},0)),(\rho_y,(0,0,\frac{1}{2}))] \oplus \mathbf{Z}_2^c$
FCC	24	$\mathbb{O} \oplus \mathbf{Z}_2^c$
		$\widetilde{\mathbb{O}}[(\rho_x,\frac{1}{2}\ell_1+\frac{1}{2}\ell_3),(\rho_y,\frac{1}{2}\ell_3)] \oplus \mathbf{Z}_2^c$
		$\widetilde{\mathbf{D}_4} \oplus \mathbf{Z}_2^c$
		$\widetilde{\mathbf{D}_4}^a \oplus \mathbf{Z}_2^c$
		$\widetilde{\mathbf{D}_3} \oplus \mathbf{Z}_2^c$
	48	$\mathbb{O} \oplus \mathbf{Z}_2^c$
		$\widetilde{\mathbb{O}}[(\rho_x,\frac{1}{2}\ell_2),(\rho_y,\frac{1}{2}\ell_3)] \oplus \mathbf{Z}_2^c$
		$\widetilde{\mathbb{O}}[(\rho_x,\frac{1}{2}\ell_1+\frac{1}{2}\ell_3),(\rho_y,\frac{1}{2}\ell_3)] \oplus \mathbf{Z}_2^c$
		$\widetilde{\mathbb{O}}[(\rho_x,\frac{1}{2}\ell_1+\frac{1}{2}\ell_2+\frac{1}{2}\ell_3),(\rho_y,0)] \oplus \mathbf{Z}_2^c$
BCC	24 Type 1 (case 1)	$\mathbb{O} \oplus \mathbf{Z}_2^c$
		$\widetilde{\mathbb{O}}[(\rho_x,\frac{1}{2}\ell_3),(\rho_y,\frac{1}{2}\ell_3)] \oplus \mathbf{Z}_2^c$
	24 Type 1 (case 2)	$\mathbb{O} \oplus \mathbf{Z}_2^c$
		$\widetilde{\mathbb{O}}[(\rho_x,\frac{1}{2}\ell_3),(\rho_y,\frac{1}{2}\ell_3)]$
		$\widetilde{\mathbb{T}} \oplus \mathbf{Z}_2^c$
		$\widetilde{\mathbf{D}}_4^a \oplus \mathbf{Z}_2^c$
		$\widetilde{\mathbf{D}}_4^b \oplus \mathbf{Z}_2^c$
	24 Type 2	$\mathbb{O} \oplus \mathbf{Z}_2^c$
		$\widetilde{\mathbb{O}}[(\rho_x,\frac{1}{2}\ell_3),(\rho_y,\frac{1}{2}\ell_3)] \oplus \mathbf{Z}_2^c$
		$\widetilde{\mathbb{O}}[(\rho_x,\frac{1}{2}\ell_3),(\rho_y,0)]$
	48	$\mathbb{O} \oplus \mathbf{Z}_2^c$
		$\widetilde{\mathbb{O}}[(\rho_x,\frac{1}{2}\ell_3),(\rho_y,\frac{1}{2}\ell_3)] \oplus \mathbf{Z}_2^c$

Figure Captions

Figure 1. The projected skeleton $\mathbb{X}^p_{\mathbf{D}_3[\tau,\kappa]}$ on the SC lattice.

Figure 2. Projected skeletons for the FCC lattice. (a) $\mathbb{X}^p_{\mathbf{D}_4[\rho,\kappa_1]}$, (b) $\mathbb{X}^p_{\mathbb{T}[\tau,\rho_1]}$ and (c) $\mathbb{X}^p_{\mathbf{D}_3[\tau,\kappa]}$.

Figure 3. Projected skeletons for the BCC lattice. (a) $\mathbb{X}^p_{\mathbf{D}_3[\tau,\kappa]}$ and (b) $\mathbb{X}^p_{\mathbb{T}[\tau,\rho_1]}$.

Figures

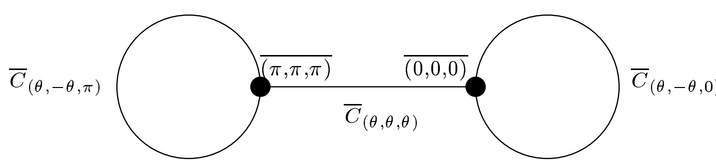
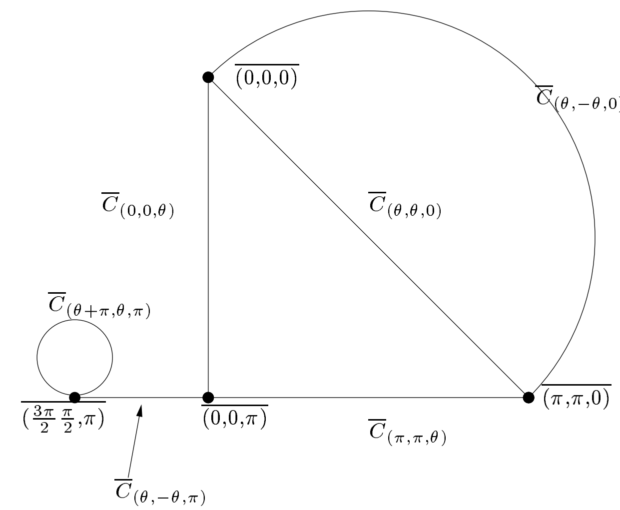
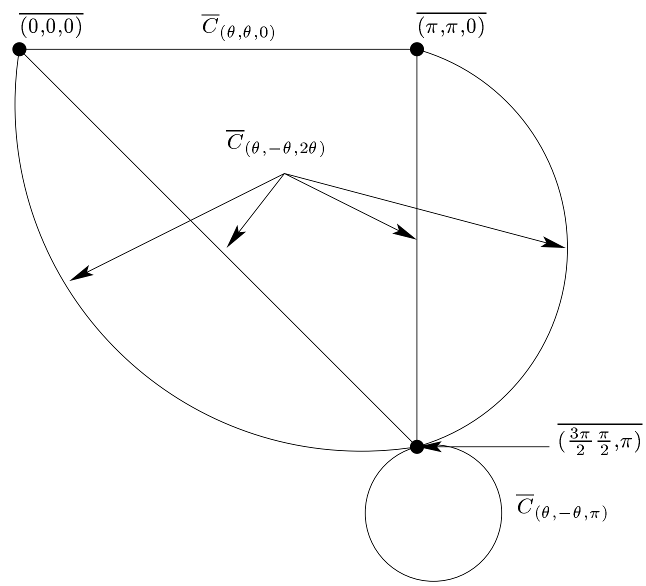


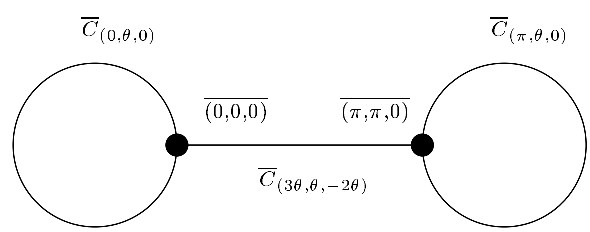
Figure 1: The projected skeleton $\mathbb{X}^p_{\mathbf{D}_3[\tau,\kappa]}$ on the SC lattice.



(a)

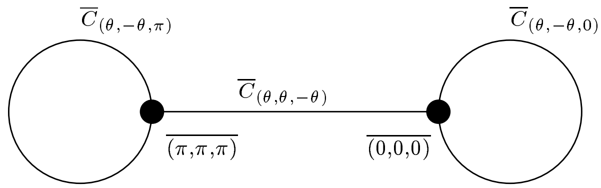


(b)

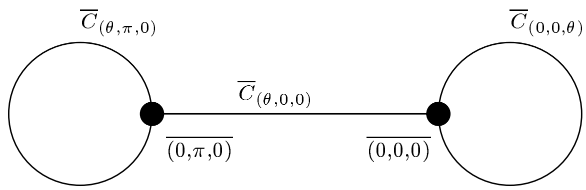


(c)

Figure 2: Projected skeletons for the FCC lattice. (a) $\mathbb{X}^p_{\mathbf{D}_4[\rho,\kappa_1]}$, (b) $\mathbb{X}^p_{\mathbb{T}[\tau,\rho_1]}$



(a) $\mathbb{X}^p_{\mathbf{D}_3[\tau,\kappa]}$



(b) $\mathbb{X}^p_{\mathbb{T}[\tau,\rho_1]}$

Figure 3: Projected skeletons for the BCC lattice. (a) $\mathbb{X}^p_{\mathbf{D}_3[\tau,\kappa]}$ and (b) $\mathbb{X}^p_{\mathbb{T}[\tau,\rho_1]}$.