Dynamical Systems: An International Journal
Vol. 00, No. 00, January 2005, 1–33

Partial classification of heteroclinic behaviour associated with the perturbation of hexagonal planforms

M J Parker, Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK
I N Stewart, Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK
M G M Gomes, Instituto Gulbenkian de Ciencia, Rua da Quinta Grande 6, 2780-156 Oeiras, Portugal

(January 2005)

Physical systems often exhibit pattern-forming instabilities. Equivariant bifurcation theory is often used to investigate the existence and stability of spatially doubly periodic solution with respect to the hexagonal lattice. Previous studies have focused on the six- and twelve-dimensional representation of the hexagonal lattice where the symmetry of the model is perfect. Here the perturbation of the group orbits of translation free axial planforms in the six- and twelve-dimensional representations is considered. This problem is studied via the abstract action of the symmetry group of the perturbation on the group orbit of the planform. A partial classification for the behaviour of the group orbits is obtained, showing the existence of homoclinic and heteroclinic cycles between equilibria.

Keywords: Forced Symmetry-Breaking; Hexagonal Planforms; Heteroclinic Cycles.

2000 Mathematics Subject Classification: 37G40, 37C29, 22E99.

1 Introduction

Many natural phenomena exhibit spatially periodic states. These systems are generally modelled by partial differential equations (PDEs) that are invariant under translations, rotations and reflections of an infinite plane. Such systems include the Navier–Stokes equations, models of hallucination patterns in the primary visual cortex and Rayleigh–Bénard convection [1].

Spatially periodic solutions are found directly by restricting the equations to a planar lattice. The symmetries of the restricted system consist of a 2-torus of translation symmetries modulo the lattice and the *holohedry* of rotations and reflections that preserve the lattice. The bifurcation problem then has compact symmetry group. Standard techniques [2] enumerate all irreducible

E-mail: martynp@maths.warwick.ac.uk, Telephone: +44 024 76 528337

representations and axial subgroups. The equivariant branching lemma [2–4] guarantees the existence of solutions with axial symmetry.

This paper considers those spatially periodic time-independent solutions, the so-called *planforms*, supported by the hexagonal lattice. A complete study of this lattice was performed group-theoretically by Dionne and Golubitsky [5]. The symmetry group of the hexagonal lattice is $\Gamma = \mathbf{D}_6 \dot{+} \mathbf{T}^2$, where $\dot{+}$ denotes semi-direct sum [2, p. 144]. Dionne and Golubitsky [5] consider only the so-called translation-free irreducible representations of Γ. A representation is said to be *translation free* if the only translation that acts trivially is the identity translation. These irreducible representations are on $\mathbb{C}^3$ and $\mathbb{C}^6$, of which there is a countable infinity of representations of Γ on $\mathbb{C}^6$. The authors also derive the symmetry of the primary solutions in the six- and twelve-dimensional representations guaranteed to exist by the equivariant branching lemma—the axial solutions. This classification proves the existence of two solutions: 'hexagons' and 'superhexagons'. The precise form of superhexagons depends on the exact representation on $\mathbb{C}^6$.

The modelling process often introduces additional symmetries that are only approximately present in the real physical system. Provided the real system is only a small perturbation of the idealized mathematical model, we would expect the ideal model to capture 'key features' of the real system. Here we investigate how the solutions of the ideal Γ-equivariant model are affected when perturbed to a system with symmetry group Δ, where Δ is a subgroup of Γ. This process is called *forced symmetry-breaking*, or sometimes *system* or *explicit symmetry-breaking*. There have been several different approaches to this problem, either focusing on the persistence of equilibria [6] or periodic orbits [7–10], perturbations of homoclinic cycles has also been considered [11]. For a general class of problems the approach adopted by these authors is not appropriate. In this paper we focus on the formulation of Lauterbach and Roberts [12] and Lauterbach *et al* [13]. Using this formalism we investigate the dynamics resulting from forced symmetry-breaking of hexagons and superhexagons. More precisely, let

$$\dot{x} = f(x) \tag{1}$$

be a Γ-equivariant system of differential equations. Suppose that (1) has an equilibrium x_0. Equivariance implies that $X_0 = \Gamma x_0$ is a group orbit of equilibria. Generically, this group orbit is a normally hyperbolic manifold [14]. Consider a small perturbation of (1)

$$\dot{x} = f(x) + \varepsilon g(x), \tag{2}$$

where ε is small and g is Δ-equivariant, where $\Delta \subseteq \Gamma$ is a Lie subgroup.

When ε is sufficiently small, the normal hyperbolicity of X_0 implies there is a perturbed flow-invariant manifold X_ε for (2) that is diffeomorphic to X_0 [12]. The dynamics of the perturbed flow can be more complex than just equilibria, indeed heteroclinic cycles can result.

For each representation of Γ we classify the (perturbed) equilibria and heteroclinic connections that are forced by symmetry to occur on X_ε. This leads to a geometric object called the *skeleton*. The quotient of the skeleton by the Δ-action gives the *projected skeleton*. The projected skeleton classifies all the 'different' heteroclinic connections and equilibria on the skeleton. For several subgroups Δ it is possible to exhibit Δ-equivariant perturbations that give heteroclinic or homoclinic cycles for the perturbed flow. However, the complexities of the invariant theory mean that a general classification is not possible. Notwithstanding this, for one subgroup conditions are determined on the lowest order Taylor coefficients of the perturbation that guarantee the existence of a homoclinic cycle for the perturbed flow.

The corresponding problem on the square lattice has been considered previously. Hou and Golubitsky [15] consider a perturbation of a square planform and prove that asymptotically stable heteroclinic cycles exist for an open set of perturbations. More recently, Parker *et al* [16] extended these results. In particular they consider a general class of perturbations that can exhibit heteroclinic connections between equilibria in both the four- and eight-dimensional representations of $\mathbf{D}_4 \dot{+} \mathbf{T}^2$: the symmetry group of the square lattice.

This paper is organized as follows. Section 2 provides the necessary background on forced symmetry-breaking, in particular its formulation as a purely algebraic problem. We then discuss the standard equivariant bifurcation theory required to analyse hexagonal planforms. Section 3.1 contains our analysis of the six-dimensional representation. Here we characterize the behaviour of the flows on the perturbed group orbit. In particular, all equilibria and heteroclinic connection forced to exist by the residual symmetry are found. The twelve-dimensional representation is considered in Section 3.3. It is shown that heteroclinic cycles are guarantee to exist for a certain perturbation of the group orbit. Section 4 contains our conclusions.

2 Problem Formulation

This section summaries the main points of forced symmetry-breaking of group orbits of steady states and symmetry-breaking bifurcations of Euclidean-invariant PDEs. For a more detailed discussion see [5, 12, 13, 17].

2.1 *Forced Symmetry-Breaking*

Let X be a smooth finite-dimensional manifold. Let Γ be a compact Lie group acting smoothly on X:

$$\Gamma \times X \mapsto X, \quad (\gamma, x) \to \gamma x.$$

Then Lauterbach and Roberts [12, Proposition 1.1] prove:

THEOREM 2.1. *Let Γ be a compact Lie group acting smoothly on a finite-dimensional smooth manifold X. Let f be a Γ-equivariant vector field on X and suppose that Φ_f is the flow on X corresponding to f. Let $\widetilde{X} \subset X$ be a compact submanifold, invariant under the flow Φ_f and the action of Γ. Suppose that $\widetilde{X}$ is normally hyperbolic. Let $\Delta \subset \Gamma$ be a subgroup of Γ. Let g be a Δ-equivariant vector field on X. Let Φ_g be the flow on X corresponding to g. Suppose that $\|f - g\| < \varepsilon$. Then, if ε is sufficiently small, there exists a unique manifold $\widetilde{X}_\varepsilon$ near to $\widetilde{X}$, invariant under the flow Φ_g. Moreover, there exist a Δ-equivariant diffeomorphism $\Theta : \widetilde{X} \to \widetilde{X}_\varepsilon$.*

We call Theorem 2.1 the equivariant persistence theorem. This nomenclature is non-standard, although we have used it before to refer to this theorem [16,17]. In our applications $\widetilde{X}$ is a manifold of solutions to an equivariant bifurcation problem (with compact symmetry group), which generically is a normally hyperbolic manifold, see Field [14].

Let $x \in X$. The *isotropy subgroup of* x is the subgroup of Γ defined by

$$\Sigma_x = \{\gamma \in \Gamma : \gamma x = x\}.$$

The *orbit of* x is the set

$$\Gamma x = \{\gamma x : \gamma \in \Gamma\}.$$

All elements on the same group orbit have isotropy subgroup conjugate to Σ_x. It is well known that Γx is Γ-equivariantly diffeomorphic to the homogeneous space Γ/Σ_x [2]. Here $\Gamma/\Sigma_x = \{\gamma\Sigma_x : \gamma \in \Gamma\}$ is the space of left cosets.

***Group Action on* $\boldsymbol{\Gamma/\Sigma}$.** Let Σ be an isotropy subgroup of Γ. Let Δ be a subgroup of Γ. Define an action of Δ on the space Γ/Σ by

$$\begin{aligned} \Delta \times (\Gamma/\Sigma) &\to (\Gamma/\Sigma), \\ (\delta, \gamma\Sigma) &\to \delta\gamma\Sigma. \end{aligned}$$

Since the orbit $\widetilde{X} = \Gamma x$ is Γ-equivariantly diffeomorphic to Γ/Σ, there is an induced action of Δ on the manifold $\widetilde{X}$. Suppose g is a Δ-equivariant vector field sufficiently close to f. Then there exists a smooth, Δ-invariant and flow-invariant manifold $\widetilde{X}_\varepsilon$ close to $\widetilde{X}$. Moreover, there exists a Δ-equivariant diffeomorphism $\Theta : \widetilde{X}_\varepsilon \to \widetilde{X}$ [12]. The Δ-action on $\widetilde{X}$ induces an action of Δ on $\widetilde{X}_\varepsilon$. Furthermore, Δ-equivariant vector fields (or flows) on $\widetilde{X}$ are the restriction of a Δ-equivariant perturbation of a Γ-equivariant vector field on X, see Lauterbach and Roberts [12, Proposition 1.3].

Let Δ' be a subgroup of Δ. Then the *fixed-point subset of* Δ' is defined by

$$\mathrm{Fix}_{\Gamma/\Sigma}(\Delta') = \{x \in \Gamma/\Sigma \big| \delta x = x \text{ for all } \delta \in \Delta'\}.$$

The fixed-point subset $\mathrm{Fix}_{\Gamma/\Sigma}(\Delta')$ is invariant under Δ-equivariant flows. Define the *isotropy subgroup of* $x \in \Gamma/\Sigma$ by

$$\mathrm{Stab}(x) = \{\delta \in \Delta \big| \delta x = x\}.$$

Let $x \in \Gamma/\Sigma$. Let $C = C(x)$ be the connected component of $\mathrm{Fix}(\mathrm{Stab}(x)) \subseteq \Gamma/\Sigma$ which contains x. Let $\mathscr{C}_\Delta$ be the collection of those Cs which are homeomorphic to $\{0\}$ or S^1. If $C \in \mathscr{C}_\Delta$ is homeomorphic to S^1, then we call C a *connecting orbit.* The set $\mathscr{C}_\Delta$ is invariant under the Δ-action.

Definition 2.2. Let

$$\mathbb{X}_\Delta = \bigcup_{C \in \mathscr{C}_\Delta} C \subset \Gamma/\Sigma.$$

The set $\mathbb{X}_\Delta$ is called the *skeleton of* Γ/Σ *with respect to* Δ.

To save cumbersome language we shall use the term *skeleton* when the context is clear. A Δ-equivariant flow on Γ/Σ induces a Δ-equivariant flow on $\mathbb{X}_\Delta$. $\mathbb{X}_\Delta$ is a stratified manifold, in fact the strata are flow-invariant. Let $x \in \mathbb{X}_\Delta$. Let $S(x)$ be the connected component of the stratum containing x. Define $\mathscr{S}_\Delta = \{S(x) \big| x \in \mathbb{X}_\Delta\}$. Define flow-invariant subsets of $\mathbb{X}_\Delta$ as follows: given the set $\mathscr{S}_\Delta$ define

$$E_{(\Delta,\Gamma/\Sigma)} = \{S \in \mathscr{S}_\Delta \big| S \text{ is homeomorphic to } \{0\}\},$$
$$H_{(\Delta,\Gamma/\Sigma)} = \{S \in \mathscr{S}_\Delta \big| S \text{ is homeomorphic to } \mathbb{R}\}.$$

Since Γ/Σ is diffeomorphic to $\widetilde{X}$ we also write $E_{(\Delta,\widetilde{X})} \equiv E_{(\Delta,\Gamma/\Sigma)}$ and $H_{(\Delta,\widetilde{X})} \equiv H_{(\Delta,\Gamma/\Sigma)}$; it is sometimes more convenient to use this notation.

Symmetry Properties of the Skeleton. Let $C \in \mathscr{C}_\Delta$ or $\mathscr{J}_\Delta$. The *pointwise isotropy of* C is defined by

$$\mathrm{stab}(C) = \{\delta \in \Delta \big| \delta x = x \text{ for all } x \in C\}.$$

There is an induced action of Δ on $\mathscr{C}_\Delta$ given by permutation. Given $C \in \mathscr{C}_\Delta$ the *setwise isotropy of* C is

$$\mathrm{Stab}(C) = \{\delta \in \Delta \big| \delta C = C\}.$$

The pointwise isotropy, $\mathrm{stab}(C)$, of $C \in \mathscr{C}_\Delta$ is normal in the setwise isotropy $\mathrm{Stab}(C)$. Thus we can form the quotient group:

$$\mathrm{Stab}(C)/\mathrm{stab}(C).$$

This group has a natural action on C. Let $C \in \mathscr{C}_\Delta$ be a connecting orbit and $x \in C$. Suppose that x is a fixed-point of some element of $\mathrm{Stab}(C)$, then x is a *knot relative to* C. Note that a knot must be an element of $E_{(\Delta,\Gamma/\Sigma)}$, however the converse is clearly false, it can be false even for those equilibria that lie on connecting orbits, see Remark *1*.

Let $H_{(\Delta,\Gamma/\Sigma)}(C)$ denote the set $\{h \in H_{(\Delta,\Gamma/\Sigma)} \big| h \subset C\}$, then Lauterbach *et al* [13] prove:

THEOREM 2.3. *Let* Δ *act on the sets* $H_{(\Delta,\Gamma/\Sigma)}$, $E_{(\Delta,\Gamma/\Sigma)}$ *and* $\mathscr{C}_\Delta$. *Then we have:*

(i) Given $e \in E_{(\Delta,\Gamma/\Sigma)}$ *or* $h \in H_{(\Delta,\Gamma/\Sigma)}$. *Then*

$$\mathrm{Stab}(e)/\mathrm{stab}(e) = \mathrm{Stab}(h)/\mathrm{stab}(h) = \mathbf{1}$$

the trivial group.

(ii) Given $C \in \mathscr{C}_\Delta$ *with* $H_{(\Delta,\Gamma/\Sigma)}(C) \neq \emptyset$, *then there are the following alternatives:*

a) Suppose $\mathrm{Stab}(C)$ *contains* $m \in \mathbb{N}$ *orientation reversing elements; then* $\mathrm{Stab}(C)/\mathrm{stab}(C) = \mathbf{D}_m$. *We use the convention* $\mathbf{D}_1 = \mathbf{Z}_2$. *The group* $\mathrm{Stab}(C)/\mathrm{stab}(C)$ *acts as the group of* m *reflections about axes through opposite knots and* $m-1$ *nontrivial rotations.*

b) Suppose $\mathrm{Stab}(C)$ *contains no orientation reversing elements; then we have* $\mathrm{Stab}(C)/\mathrm{stab}(C) = \mathbf{Z}_m$ *for some* $m \in \mathbb{N}$. *We use the convention* $\mathbf{Z}_1 = \mathbf{1}$. *The group* $\mathrm{Stab}(C)/\mathrm{stab}(C)$ *acts as rotations on* C.

The rotations and reflections preserve $H_{(\Delta,\Gamma/\Sigma)}(C)$.

Furthermore, if $H_{(\Delta,\Gamma/\Sigma)}(C) \neq \emptyset$, then the knots relative to C always occur in pairs on $\mathbb{X}_\Delta$, dividing C into two connected components with the same number of edges in $H_{(\Delta,\Gamma/\Sigma)}(C)$ on each component. There are two possible types of behaviour (Lauterbach *et al* [13]):

(i) Suppose there are no knots relative to C. Then $\mathrm{Stab}(C)/\mathrm{stab}(C)$ is isomorphic to some $\mathbf{Z}_m$ (with $\mathbf{Z}_1 = \mathbf{1}$) and acts on $H_{(\Delta,\Gamma/\Sigma)}(C)$ by rotations on C.

(ii) Suppose that there are knots relative to C. Then $\mathrm{Stab}(C)/\mathrm{stab}(C)$ is isomorphic to some $\mathbf{D}_m$ (with $\mathbf{D}_1 = \mathbf{Z}_2$). This group acts as $m-1$ non-trivial rotations and m reflections on $H_{(\Delta,\Gamma/\Sigma)}(C)$. The pairs of opposite knots give the axes of reflections. In particular, m is the number of knots relative to C.

Vector Fields on the Skeleton. Define

$$\pi : \mathbb{X}_\Delta \to \Delta\backslash\mathbb{X}_\Delta \quad \text{by} \quad \pi(x) = {}_\Delta[x],$$

where ${}_\Delta[x]$ is the equivalence class of all points x which are members of the same group orbit. Let $\mathbb{X}^p_\Delta = \Delta\backslash\mathbb{X}_\Delta$. Then $\mathbb{X}^p_\Delta$ is called the *projected skeleton.*

By the smooth lifting theorem of Schwarz (see Lauterbach and Roberts [12]), this map is surjective. So every stratum preserving smooth vector field on $\Gamma\backslash X$ lifts to a smooth Γ-equivariant vector field on X, and any flow on the projected skeleton lifts to a Δ-equivariant flow on the skeleton, giving the following existence result, see Lauterbach *et al* [13, Corollary 3.32]:

THEOREM 2.4. *Let $h \in H_{(\Delta,\Gamma/\Sigma)}$. Then there exist Δ-equivariant vector fields and corresponding flows Φ on $\mathbb{X}_\Delta$ such that h is a heteroclinic connection of Φ connecting equilibria in $E_{(\Delta,\Gamma/\Sigma)}$.*

Perturbed Flows. We make an addition assumption on $\widetilde{X}$; we assume that $\widetilde{X}$ has an inner product structure. In most applications this condition is satisfied. Let $\Theta_\varepsilon : \widetilde{X} \to \widetilde{X}_\varepsilon$ be the Δ-equivariant diffeomorphism given in Theorem 2.1. Then Θ_0 is the identity map. Let $e_j \in E_{(\Delta,\widetilde{X})}$ and define $e^\varepsilon_j = \Theta_\varepsilon(e_j)$. Let $h_j \in H_{(\Delta,\widetilde{X})}$ and define $h^\varepsilon_j = \Theta_\varepsilon(h_j)$. Define $E^\varepsilon_{(\Delta,\widetilde{X}_\varepsilon)} = \{\Theta_\varepsilon(e) \big| e \in E_{(\Delta,\widetilde{X})}\}$ and $H^\varepsilon_{(\Delta,\widetilde{X}_\varepsilon)} = \{\Theta_\varepsilon(h) \big| h \in H_{(\Delta,\widetilde{X})}\}$. Then define

$$\mathbb{X}^\varepsilon_\Delta = E^\varepsilon_{(\Delta,\widetilde{X}_\varepsilon)} \cup H^\varepsilon_{(\Delta,\widetilde{X}_\varepsilon)} \subseteq \widetilde{X}_\varepsilon.$$

We call $\mathbb{X}^\varepsilon_\Delta$ the *perturbed skeleton.*

Let $h \in H_{(\Delta,\widetilde{X})}$ and suppose that h connects two points e_i and $e_j \in E_{(\Delta,\widetilde{X})}$. By definition, there exists a subgroup Δ' of Δ such that h is contained in the fixed-point subset $\mathrm{Fix}_{\widetilde{X}}(\Delta')$, which is diffeomorphic to S^1. There exists a smooth function $\gamma^* : [0, 2\pi] \to \Gamma$ and an injective smooth function ω :

$[0, 2\pi] \to S^1$ such that $\omega(t) = \gamma^*(t)x$ is a nondegenerate parametrisation of the fixed-point subset $\mathrm{Fix}_{\widetilde{X}}(\Delta')$. In addition the following hold:

$$h = \{\omega(t) \big| t \in (0, t^*)\}, \quad 0 < t^* \leq 2\pi; \quad \omega(0) = e_i \text{ and } \omega(t^*) = e_j.$$

Let g be a Δ-equivariant vector field. Let $h \in H_{(\Delta,\widetilde{X})}$. Let ω be a parametrisation of h. Let

$$v(t) = \frac{\mathrm{d}}{\mathrm{d}t}\omega(t).$$

Then

$$\mathcal{T}_\omega(t) = \frac{v(t)}{\|v(t)\|},$$

is the unit tangent vector to $\omega(t)$. The flow along the connection h is given by:

$$\mathcal{F}^g_h(t) = \langle g(\omega(t)), \mathcal{T}_\omega(t)\rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\widetilde{X}$. We call the function $\mathcal{F}^g_h$ the *flow formula* [18]. If g or h are clear from context then we omit them.

Definition 2.5. Let $h \in H_{(\Delta,\widetilde{X})}$ and suppose h connects the elements e_i and $e_j \in E_{(\Delta,\widetilde{X})}$. Let $\omega : [0, t^*] \to \widetilde{X}$ parametrise h, with $\omega(0) = e_i$ and $\omega(t^*) = e_j$. Let g be a Δ-equivariant vector field. Then g is an *admissible perturbation for the connection* h if $\mathcal{F}^g_h(t) \neq 0$ for all $t \in (0, t^*)$. If g is an admissible perturbation for all connections $h \in H_{(\Delta,\widetilde{X})}$, then we call g an *admissible perturbation* and the corresponding flow an *admissible flow*.

Definition 2.6. Let $h \in H_{(\Delta,\widetilde{X})}$ and suppose h connects the elements e_i and $e_j \in E_{(\Delta,\widetilde{X})}$. Let $\omega : [0, t^*] \to \widetilde{X}$ parametrise h, with $\omega(0) = e_i$ and $\omega(t^*) = e_j$. Let g be a Δ-equivariant vector field. Suppose there exist $t_1 \in (0, t^*)$ such that $\mathcal{F}^g_h(t_1) = 0$ and $(d/dt)\mathcal{F}^g_h(t_1) \neq 0$. Then g is called a *simple degenerate perturbation*. Any other perturbation g which is not admissible and not a simple degenerate perturbation is called a *degenerate perturbation*.

Let $h^\varepsilon \in H^\varepsilon_{(\Delta,\widetilde{X}_\varepsilon)}$. Suppose h^ε connects e^ε_i and $e^\varepsilon_j \in E^\varepsilon_{(\Delta,\widetilde{X}_\varepsilon)}$. By definition there exist $h \in H_{(\Delta,\widetilde{X})}$ and e_i, $e_j \in E_{(\Delta,\widetilde{X})}$ such that $h^\varepsilon = \Theta_\varepsilon(h)$, $e^\varepsilon_i = \Theta_\varepsilon(e_i)$ and $e^\varepsilon_j = \Theta_\varepsilon(e_j)$. Let $\omega : [0, t^*] \to \widetilde{X}$ be a parametrisation of h. Then the function $\omega^\varepsilon : [0, t^*] \to \widetilde{X}_\varepsilon$ defined by $\omega^\varepsilon(t) = \Theta_\varepsilon(\omega(t))$ is a nondegenerate parametrisation of h^ε such that $\omega^\varepsilon(0) = e_i^\varepsilon$ and $\omega^\varepsilon(t^*) = e_j^\varepsilon$. Thus we have a parametrisation for the connections in $H^\varepsilon_{(\Delta,\widetilde{X}_\varepsilon)}$. Let

$$v^\varepsilon(t) = \frac{\mathrm{d}}{\mathrm{d}t}\omega^\varepsilon(t),$$

and

$$\mathcal{T}_{\omega^\varepsilon}(t) = \frac{v^\varepsilon(t)}{\|v^\varepsilon(t)\|}.$$

Define

$$\mathcal{F}_{h^\varepsilon}^{\varepsilon,g}(t) = \langle g(\omega^\varepsilon(t)), \mathcal{T}_{\omega^\varepsilon}(t)\rangle \, .$$

Then $\mathcal{F}_{h^\varepsilon}^{\varepsilon,g}(t)$ gives the flow on the connection h^ε on the perturbed skeleton. The behaviour of the flow on the perturbed skeleton is related to the flow formula for the flow on the skeleton via the following unfolding theorem [17].

THEOREM 2.7. *Let g be a Δ-equivariant vector field and $h^\varepsilon \in H^\varepsilon_{(\Delta,\widetilde{X}_\varepsilon)}$. Suppose h^ε connects e_i^ε and $e_j^\varepsilon \in E^\varepsilon_{(\Delta,\widetilde{X}_\varepsilon)}$. Let $\Theta_\varepsilon : \widetilde{X} \to \widetilde{X}_\varepsilon$ be the Δ-equivariant diffeomorphism given in Theorem 2.1. Let $h \in H_{(\Delta,\widetilde{X})}$, e_i and $e_j \in E_{(\Delta,\widetilde{X})}$ be chosen so that $h^\varepsilon = \Theta_\varepsilon(h)$, $e_i^\varepsilon = \Theta_\varepsilon(e_j)$ and $e_j^\varepsilon = \Theta_\varepsilon(e_j)$. Let $\omega : [0,t^*] \to \widetilde{X}$ parametrise h. Suppose there exists $t_0 \in (0,t^*)$ such that $\mathcal{F}_h^g(t_0) = \ldots = (d^{(n-1)}/dt^{(n-1)})\mathcal{F}_h^g(t_0) = 0$ and $(d^{(m)}/dt^{(m)})\mathcal{F}_h^g(t_0) \neq 0$ for all $m \geq n$. Then for sufficiently small ε the behaviour of the function $\mathcal{F}_{h^\varepsilon}^{\varepsilon,g}(t)$ in a neighbourhood of t_0 is given by the universal unfolding*

$$a_0 + a_1 t + \ldots + a_{n-2}t^{n-2} + a_n t^n,$$

of $a_n t^n$, where $a_j \in \mathbb{R}$.

This theorem states that if the flow formula $\mathcal{F}_h^g(t)$ has either no zeros or simple zeros along the connection h, then the same is true for the flow along h^ε. If the flow formula $\mathcal{F}_h^g(t)$ is degenerate along h then their behaviour on h^ε cannot be known explicitly, only a finite list of possibilities given.

2.2 *PDEs with Euclidean Symmetry*

Consider a parameterized family of PDEs

$$\frac{\partial}{\partial t}u(x,t) = F\left(u(x,t),\lambda\right) \tag{3}$$

where $F : \mathcal{X} \times \mathbb{R} \to \mathcal{Y}$ is a nonlinear operator between suitable function spaces $\mathcal{X}$ and $\mathcal{Y}$, and $\lambda \in \mathbb{R}$ is a bifurcation parameter. The function $u : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^n$ is a function in $\mathcal{X}$ of a spatial variable $x \in \mathbb{R}^2$ and time t.

The system of PDEs (3) is assumed to have Euclidean symmetry. In addition, we assume that there is an Euclidean-invariant time-independent solution of (3) for all values of λ. Without loss of generality we assume that this spatially uniform solution corresponds to $u = 0$, that is

$$F(0,\lambda) = 0,$$

for all λ. Furthermore, we assume that this solution is stable for $\lambda < 0$, unstable for $\lambda > 0$ and that $\lambda = 0$ corresponds to a steady-state symmetry-breaking bifurcation point.

To overcome the problems introduced by the non-compactness of the Euclidean group $\mathbf{E}(2)$ we seek spatially periodic, time-independent solutions $u(x,t)$ to (3). Let $\mathcal{L}$ be the planar hexagonal lattice generated by $\ell_1 = (2/\sqrt{3},0)$, $\ell_2 = (1/\sqrt{3},1)$; that is,

$$\mathcal{L} = \{n_1\ell_1 + n_2\ell_2 \big| n_1,\ n_2 \in \mathbb{Z}\}.$$

The symmetry group of $\mathcal{L}$ is $\Gamma = \mathbf{D}_6 \dot{+} \mathbf{T}^2$. Here $\mathbf{D}_6$ is the dihedral group of order twelve generated by a rotation ρ by $\pi/3$ anticlockwise and a reflection κ in the line $x = 0$. We say that a function u is $\mathcal{L}$-periodic if $u(x+\ell) = u(x)$ for all $\ell \in \mathcal{L}$. The subspace $\mathcal{X}_{\mathcal{L}} \subset \mathcal{X}$ of $\mathcal{L}$-periodic functions is $\mathcal{X}_{\mathcal{L}} = \big\{f \in \mathcal{X} \big| f(x+\ell) = f(x) \text{ for all } \ell \in \mathcal{L}\big\}$. The group Γ is the largest subgroup of $\mathbf{E}(2)$ that preserves $\mathcal{X}_{\mathcal{L}}$; that is, $\gamma\mathcal{X}_{\mathcal{L}} \subseteq \mathcal{X}_{\mathcal{L}}$ for all $\gamma \in \Gamma$.

The dual lattice $\mathcal{L}^*$ of $\mathcal{L}$ is the set

$$\mathcal{L}^* = \left\{\mathbf{k} \in \mathbb{R}^2 \big| x \mapsto e^{2\pi i \mathbf{k}\cdot x} \text{ is } \mathcal{L}\text{−periodic}\right\}.$$

Define $\mathbf{k}_1 = (1,0)$ and $\mathbf{k}_2 = (\frac{1}{2}\sqrt{3}, -\frac{1}{2})$. Then the dual lattice to $\mathcal{L}$ is given by

$$\mathcal{L}^* = \{n_1\mathbf{k}_1 + n_2\mathbf{k}_2 \big| n_1,\ n_2 \in \mathbb{Z}\}.$$

We assume that a function $u \in \mathcal{X}_{\mathcal{L}}$ can be written in the form

$$u(x,t) = \sum_{j=1}^{s} z_j e^{2\pi \mathrm{i} \mathbf{K}_j \cdot x} + c.c., \tag{4}$$

where $z_j \in \mathbb{C}$ and *c.c.* denotes complex conjugate. The circle $|\mathbf{K}_j| = k_c$ in the two-dimensional **k**-space is called the *critical circle.* The dimension of the bifurcation problem depends on the number of vectors $\mathbf{k} \in \mathcal{L}^*$ which lie on the critical circle. In this case $k_c = |\mathbf{K}_j| = \sqrt{\alpha^2 + \beta^2 - \alpha\beta}$ for integers α and β [19]. Dionne and Golubitsky [19] show that $s = 3$ or 6. The case $s = 3$ occurs, for example, when $k_c = 1$, and $(\alpha, \beta) = (1, 0)$. We have $s = 6$ when, for example, $k_c = \sqrt{7}$ and $(\alpha, \beta) = (3, 2)$. The PDEs, by a Liapunov–Schmidt reduction [20] or restriction to the centre manifold [21], give a system of ODEs on $\mathbb{C}^s$. The representation of Γ on $\mathbb{C}^s$ is determined by its action on the complex amplitudes z_j in (4). All representations are translation free. The problem is now in standard form; we consider the system of ODEs

$$\dot{z} = f(z, \lambda), \tag{5}$$

where $f : \mathbb{C}^s \times \mathbb{R} \to \mathbb{C}^s$ is Γ-equivariant, $f(0, \lambda) = 0$ and the Jacobian matrix at the bifurcation point $(\mathrm{d}f)_{0,0}$, is the zero matrix. The results of Dionne and Golubitsky [19] show there exist two translation free axial solutions, one in the six-dimensional representation ($s = 3$) and one in the twelve-dimensional representation ($s = 6$). An axial solution is *translation free* its isotropy subgroup Σ satisfies

$$\Sigma \cap \mathbf{T}^2 = \mathbf{1}.$$

Six-Dimensional Representation. The representation of Γ on $\mathbb{C}^3$ corresponds to the following action. Choose coordinates $z = (z_1, z_2, z_3)$ on $\mathbb{C}^3$. The action of Γ is generated by

$$\begin{aligned} \rho(z_1, z_2, z_3) &= (\overline{z_2}, \overline{z_3}, \overline{z_1}), \\ \kappa(z_1, z_2, z_3) &= (z_1, z_3, z_2), \\ \theta(z_1, z_2, z_3) &= (e^{\mathrm{i}\theta_1} z_1, e^{\mathrm{i}\theta_2} z_2, e^{-\mathrm{i}(\theta_1+\theta_2)} z_3), \end{aligned} \tag{6}$$

where $(\theta_1, \theta_2) \in \mathbf{T}^2$.

There are two approaches to bifurcation problems on the hexagonal. Firstly, a general Γ-equivariant bifurcation problem possess a quadratic equivariant, implying all bifurcating solutions are (locally) unstable [2, 22]. Buzano and

Golubitsky [22] use singularity theory (specifically normal forms and universal unfoldings) to analyse this degenerate problem. It is not straight forward to apply these methods to more general bifurcation problems. The second approach is computationally more straight forward. We consider a $\Gamma \oplus \mathbf{Z}_2$-equivariant bifurcation problem, where $\mathbf{Z}_2$ acts as minus the identity. The additional $\mathbf{Z}_2$ symmetry removes all even order terms from the bifurcation problem allowing standard techniques to be applied, see Golubitsky *et al.* [23]. We consider only the pure Γ-equivariant problem. The action of Γ on $\mathbb{C}^3$ has eight conjugacy classes of isotropy subgroups [22]. Of these subgroups the only translation free axial subgroup is $\mathbf{D}_6$ and we shall focus exclusively on this group. Summarising we have the following, see [19, 22]:

THEOREM 2.8. *Let $f : \mathbb{C}^3 \times \mathbb{R} \to \mathbb{C}^3$ be a Γ-equivariant bifurcation problem. Then, generically, there exists a branch of steady-state solutions bifurcating from the origin with $\mathbf{D}_6$-isotropy.*

Buzano and Golubitsky [22] show that an unfolding of the quadratic degeneracy can yield stable solutions. The solutions given by Theorem 2.8 are referred to in the literature as *hexagons*, the same convention is adopted here.

Twelve-Dimensional Representation. The representation of Γ on $\mathbb{C}^6$ corresponds to the following action, see Dionne and Golubitsky [19]. Choose coordinates $z = (z_1, z_2, z_3, z_4, z_5, z_6)$ on $\mathbb{C}^6$. Let α and β be integers which satisfy the conditions: 1) $\alpha > \beta > \alpha/2 > 0$, 2) α and β are relatively prime, 3) $\alpha + \beta$ is not a multiple of 3. Then

$$\begin{aligned} \rho(z_1, z_2, z_3, z_4, z_5, z_6) &= (\overline{z_2}, \overline{z_3}, \overline{z_1}, \overline{z_5}, \overline{z_6}, \overline{z_4}), \\ \kappa(z_1, z_2, z_3, z_4, z_5, z_6) &= (z_6, z_5, z_4, z_3, z_2, z_1), \end{aligned}$$

and

$$\begin{aligned} \theta(z_1, z_2, z_3, z_4, z_5, z_6) = (&e^{-\mathrm{i}(\alpha\theta_1 + \beta\theta_1)} z_1, e^{-\mathrm{i}((-\alpha+\beta)\theta_1 - \alpha\theta_2)} z_2, \\ &e^{-\mathrm{i}((-\beta\theta_1 + (\alpha-\beta)\theta_2)} z_3, e^{-\mathrm{i}(\alpha\theta_1 + (\alpha-\beta)\theta_2)} z_4, e^{-\mathrm{i}(-\beta\theta_1 - \alpha\theta_2)} z_5, e^{-\mathrm{i}((-\alpha+\beta)\theta_1 + \beta\theta_2)} z_6). \end{aligned} \tag{7}$$

The action of the group Γ on $\mathbb{C}^6$ has six conjugacy classes of axial subgroups, see Dionne *et al.* [5], the only translation free axial subgroup is $\mathbf{D}_6$. A general Γ-equivariant map contains a quadratic term, implying that, generically, all axial solutions are unstable at bifurcation [2, 22]. A singularity theory approach, like that taken by Buzano and Golubitsky [22] in the six-dimensional representation, is too complex. Dionne *et al* [19] show that in the degenerate problem, where the quadratic term is zero, any solution, in particular the $\mathbf{D}_6$-symmetric solution, can be stable. The authors also consider the $\Gamma \oplus \mathbf{Z}_2$-equivariant bifurcation problem. Here there are four translation free axial solutions each with non-conjugate $\mathbf{D}_6$ isotropy. These solutions are referred to as superhexagons, anti-hexagons, supertriangles and anti-triangles. It is possible for these solutions to be stable at bifurcation [19]. We shall not consider the last three cases. The above discussion gives, see [5, 19]:

THEOREM 2.9. *Let $f : \mathbb{C}^6 \times \mathbb{R} \to \mathbb{C}^6$ be a Γ-equivariant bifurcation problem. Then, generically, there exists a branch of solutions bifurcating from the origin with $\mathbf{D}_6$-isotropy. In the nondegenerate problem this branch is unstable.*

Dionne *et al* [19] show that if the degenerate problem is weakly unfolded (the coefficient of the quadratic term is made nonzero, but small), then many secondary transitions are possible along the $\mathbf{D}_6$ branch of solutions. In particular the branch can gain stability at a secondary saddle-node bifurcation [19]. Therefore, there exists a region in the parameter space where the bifurcating $\mathbf{D}_6$ symmetric solution given by Theorem 2.9 is stable. The group orbit of the solutions in the six- and twelve-dimensional representations is diffeomorphic to the 2-torus, which we denoted throughout by X_0.

3 Forced Symmetry-Breaking of Hexagonal Planforms

Generically the manifold X_0 of hexagonal solutions is normally hyperbolic. We investigate the behaviour of the group orbit X_0 when the system is perturbed with terms having $\mathbf{D}_6$, $\mathbf{D}_3[\rho^2, \kappa]$, $\mathbf{D}_3[\rho^2, \kappa\rho]$, $\mathbf{D}_2[\rho^3, \kappa]$, $\mathbf{D}_2[\rho^3, \kappa\rho]$ and $\mathbf{D}_2[\rho^3, \kappa\rho^2]$ symmetry. In particular we are interested in the existence of equilibria and heteroclinic connections on X_0, and admissible perturbations. Throughout this section Δ will denote one of the following subgroups of $\mathbf{D}_6$: $\mathbf{D}_6$, $\mathbf{Z}_6[\rho]$, $\mathbf{D}_3[\rho^2, \kappa]$, $\mathbf{D}_3[\rho^2, \kappa\rho]$, $\mathbf{D}_2[\rho^3, \kappa]$, $\mathbf{D}_2[\rho^3, \kappa\rho]$, or $\mathbf{D}_2[\rho^3, \kappa\rho^2]$. If we wish to consider one of these groups specifically then we write that group explicitly.

3.1 *Six-dimensional Representation*

Let $g : \mathbb{C}^3 \to \mathbb{C}^3$ be a Δ-equivariant mapping. Let ε be real and small. Consider the Δ-equivariant perturbation of (5), $P : \mathbb{C}^3 \times \mathbb{R}^2 \to \mathbb{C}^3$ defined by

$$P(z, \lambda, \varepsilon) = f(z, \lambda) + \varepsilon g(z),$$

where f is a Γ-equivariant vector field. Since X_0 is normally hyperbolic, the equivariant persistence theorem (Theorem 2.1) implies there exists a P- and Δ-invariant manifold X_ε which is Δ-equivariantly diffeomorphic to X_0.

3.1.1 ***Calculation of the skeletons.*** There are sixteen closed subgroups of $\mathbf{D}_6$. These groups are:

$$\begin{array}{llllll} \mathbf{D}_6, & \mathbf{Z}_6[\rho], & \mathbf{D}_3[\rho^2,\kappa], & \mathbf{D}_3[\rho^2,\kappa\rho], & \mathbf{D}_2[\rho^3,\kappa], & \mathbf{D}_2[\rho^3,\kappa\rho], \\ \mathbf{D}_2[\rho^3,\kappa\rho^2], & \mathbf{Z}_3[\rho^2], & \mathbf{Z}_2[\rho^3], & \mathbf{Z}_2[\kappa], & \mathbf{Z}_2[\kappa\rho], & \mathbf{Z}_2[\kappa\rho^2], \\ \mathbf{Z}_2[\kappa\rho^3], & \mathbf{Z}_2[\kappa\rho^4], & \mathbf{Z}_2[\kappa\rho^5], & & & \end{array}$$

and the trivial subgroup.

The action of $\mathbf{D}_6$ on $\mathbb{C}^3$ is generated by the action of ρ and κ on $\mathbb{C}^3$ given in (6). This action induces a $\mathbf{D}_6$-action on X_0 generated by:

$$\rho(\theta_1,\theta_2) = (-\theta_2, \theta_1+\theta_2),\ \kappa(\theta_1,\theta_2) = (\theta_1, -(\theta_1+\theta_2)). \tag{8}$$

The case $\mathbf{\Delta} = \mathbf{D_6}$. Let Δ' be a subgroup of Δ. The action of $\mathbf{D}_6$ in (8) implies the fixed-point subset $\mathrm{Fix}(\Delta')$ is one of the entries in Table 1. The computation can be found in detail and in a coordinate free framework in Parker [17]. To describe the skeleton it is convenient to introduce some notation. Let $C \in \mathscr{C}_\Delta$ be a connecting orbit. Choose a parametrisation $\omega : [0,1] \to X_0$ for C. We write

$$C := C_{\omega(t)}.$$

In the context of our work the range of ω is $\mathbf{T}^2$, so $\omega(t)$ is determined by two coordinates $\theta_1(t)$ and $\theta_2(t)$ for suitable functions θ_1, θ_2. Thus we write

$$C = C_{(\theta_1(t),\theta_2(t))}.$$

The entries of Table 1 imply that

$$\begin{aligned} \mathscr{C}_{\mathbf{D}_6} = \{&(0,0),(\pi,0),(0,\pi),(\pi,\pi),(2\pi/3,2\pi/3),(4\pi/3,4\pi/3), \\ &C_{(\theta,\theta)}, C_{(\theta,0)}, C_{(0,\theta)}, C_{(\theta,-\theta)}, C_{(2\theta,-\theta)}, C_{(-\theta,2\theta)}\}. \end{aligned} \tag{9}$$

Here we have slightly abused notation. For example, the singleton $(0,0)$ should be the singleton set $\{(0,0)\}$, however we use our notation since it make the presentation clearer. In each case the parameter $\theta \in [0,2\pi)$ and all singletons are points on X_0.

The skeleton $\mathbb{X}_{\mathbf{D}_6}$ can easily be determined from $\mathscr{C}_{\mathbf{D}_6}$, we omit the full details since they are not required. In Figure 1(a) we illustrate the skeleton. The figure should be interpreted in the following way. Since X_0 is diffeomorphic to a 2-torus we consider X_0 as the quotient space $[0,2\pi]^2/\sim$, where $\sim$ denotes the appropriate identification of edges of the square $[0,2\pi]^2$ (see Figure 1(g)),

Table 1. Fixed-point subsets for nontrivial subgroups of $\mathbf{D}_6$.

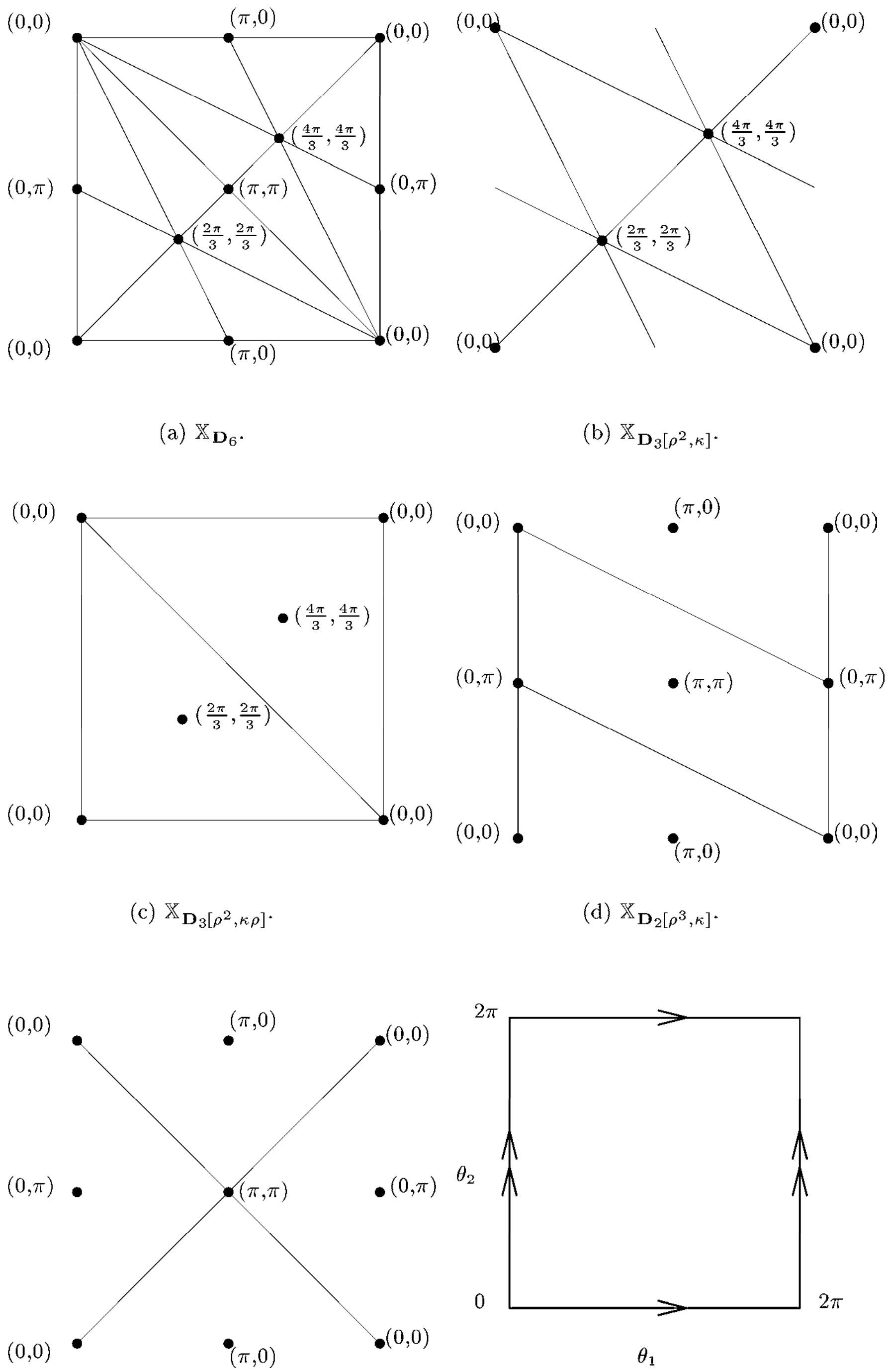
Group	Fixed-point subset(s)
$\mathbf{D}_6$	$\{(0,0)\}$
$\mathbf{Z}_6[\rho]$	$\{(0,0)\}$
$\mathbf{D}_3[\rho^2,\kappa]$	$\{(0,0)\}\cup\{(2\pi/3,2\pi/3)\}\cup\{(4\pi/3,4\pi/3)\}$
$\mathbf{D}_3[\rho^2,\kappa\rho]$	$\{(0,0)\}$
$\mathbf{D}_2[\rho^3,\kappa]$	$\{(0,0)\}\cup\{(0,\pi)\}$
$\mathbf{D}_2[\rho^3,\kappa\rho]$	$\{(0,0)\}\cup\{(\pi,\pi)\}$
$\mathbf{D}_2[\rho^3,\kappa\rho^2]$	$\{(0,0)\}\cup\{(\pi,0)\}$
$\mathbf{Z}_3[\rho^2]$	$\{(0,0)\}\cup\{(2\pi/3,2\pi/3)\}\cup\{(4\pi/3,4\pi/3)\}$
$\mathbf{Z}_2[\kappa]$	$\{(2\theta,-\theta):\theta\in[0,2\pi)\}$
$\mathbf{Z}_2[\kappa\rho]$	$\{(\theta,-\theta):\theta\in[0,2\pi)\}$
$\mathbf{Z}_2[\kappa\rho^2]$	$\{(-\theta,2\theta):\theta\in[0,2\pi)\}$
$\mathbf{Z}_2[\kappa\rho^3]$	$\{(0,\theta):\theta\in[0,2\pi)\}$
$\mathbf{Z}_2[\kappa\rho^4]$	$\{(\theta,\theta):\theta\in[0,2\pi)\}$
$\mathbf{Z}_2[\kappa\rho^5]$	$\{(\theta,0):\theta\in[0,2\pi)\}$

each figure show $[0,2\pi]^2$ before identification by $\sim$. The form of the connecting orbits in (9) and the coordinate system used on X_0 tell us which line in Figure 1(a) corresponds to which connecting orbit.

The axes used to represent the coordinates on the 2-torus are indicated in Figure 1(g). The disadvantage of visualizing the skeletons in this way is the multiple appearances of certain sets, however, this should not cause any confusion.

As discussed in Section 2.1 the skeleton $\mathbb{X}_{\mathbf{D}_6}$ is a stratified manifold, the appropriate decomposition into the sets $E_{(\mathbf{D}_6,X_0)}$ and $H_{(\mathbf{D}_6,X_0)}$ is long, and adds no clarity and is omitted at this stage. We will make reference to $E_{(\mathbf{D}_6,X_0)}$ and $H_{(\mathbf{D}_6,X_0)}$ and their elements as we require them. It is obvious, however, from Figure 1(a) that $E_{(\mathbf{D}_6,X_0)}$ consists of the four singletons (θ_1,θ_2) where $\theta_1,\theta_2=0$ or π together with $(2\pi/3,2\pi/3)$ and $(4\pi/3,4\pi/3)$. The set $H_{(\mathbf{D}_6,X_0)}$ contains those subsets of the connecting orbits which connect the equilibria.

The case when $\Delta\neq\mathbf{D}_6$. The results when $\Delta=\mathbf{D}_6$ restrict to give the remaining results. Since the groups $\mathbf{D}_2[\rho^3,\kappa]$, and $\mathbf{D}_2[\rho^3,\kappa\rho^2]$ are conjugate in $\mathbf{D}_6$ the skeletons and symmetry results are isomorphic, so the results for the $\mathbf{D}_2[\rho^3,\kappa\rho^2]$ case are omitted; the results can be derived by conjugating the results for the $\mathbf{D}_2[\rho^3,\kappa]$ case by ρ^2.



(a) $\mathbb{X}_{\mathbf{D}_6}$. (b) $\mathbb{X}_{\mathbf{D}_3[\rho^2,\kappa]}$. (c) $\mathbb{X}_{\mathbf{D}_3[\rho^2,\kappa\rho]}$. (d) $\mathbb{X}_{\mathbf{D}_2[\rho^3,\kappa]}$. (e) $\mathbb{X}_{\mathbf{D}_2[\rho^3,\kappa\rho]}$. (f) Axes.

Figure 1. The skeletons $\mathbb{X}_\Delta$.

By taking the appropriate restrictions the results in Table 1 give:

$$\begin{aligned}
\mathscr{C}_{\mathbf{D}_3[\rho^2,\kappa]} &= \{(0,0),(2\pi/3,2\pi/3),(4\pi/3,4\pi/3),C_{(\theta,\theta)},C_{(2\theta,-\theta)},C_{(-\theta,2\theta)}\},\\
\mathscr{C}_{\mathbf{D}_3[\rho^2,\kappa\rho]} &= \{(0,0),(2\pi/3,2\pi/3),(4\pi/3,4\pi/3),C_{(\theta,-\theta)},C_{(0,\theta)},C_{(\theta,0)}\},\\
\mathscr{C}_{\mathbf{D}_2[\rho^3,\kappa]} &= \{(0,0),(\pi,0),(0,\pi),(\pi,\pi),C_{(0,\theta)},C_{(2\theta,-\theta)}\},\\
\mathscr{C}_{\mathbf{D}_2[\rho^3,\kappa\rho]} &= \{(0,0),(\pi,0),(0,\pi),(\pi,\pi),C_{(\theta,\theta)},C_{(\theta,-\theta)}\}. \qquad (10)
\end{aligned}$$

Like the $\mathbf{D}_6$ case we omit the details of the skeletons. Figures 1(b–e) illustrates each skeleton. The form of the connecting orbits in (10) and the coordinate system on X_0 uniquely identifies each line in Figures 1(b–e) with a connecting orbit. Using Figures 1(b–e) we can read off the sets $E_{(\Delta,X_0)}$ and $H_{(\Delta,X_0)}$. The details of these sets are not required.

The following result is an immediate consequence of the discussion above, here Δ is any of the groups considered above.

Proposition 3.1. *Let $\Gamma = \mathbf{D}_6 \dot{+} \mathbf{T}^2$ act on $\mathbb{C}^3$ as in (6). Let f be a Γ-equivariant bifurcation problem. Let g be Δ-equivariant. Let $P(z,\lambda,\varepsilon) = f(z,\lambda)+\varepsilon g(z)$. Then there exists a steady-state solution to $f(z,\lambda)=0$ bifurcating from the origin with $\mathbf{D}_6$-isotropy. Let $X_0 \cong \mathbf{T}^2$ be the group orbit of steady states. Then for sufficiently small ε, X_0 persists to give a new P-invariant manifold X_ε which is Δ-equivariantly diffeomorphic to X_0. Moreover, there exists g such that the elements in $E_{(\Delta,X_0)}$ are equilibria for the new flow and those of $H_{(\Delta,X_0)}$ are heteroclinic connections between equilibria.*

Proof This follows from the discussion above, Theorem 2.1 and Theorem 2.4. □

This result does not specify those perturbations g that give heteroclinic connections between the equilibria in $E_{(\Delta,X_0)}$. This point is investigated further in Subsection 3.2.

3.1.2 ***Symmetry properties of the skeletons.*** We consider the restrictions placed on the skeletons by the Δ-actions on X_0 and on $\mathscr{C}_\Delta$.

The $\mathbf{D}_6$-action on X_0 induces an action on $\mathscr{C}_{\mathbf{D}_6}$. This action permutes the elements of $\mathscr{C}_{\mathbf{D}_6}$, see Table 2. The action for the subgroups of $\mathbf{D}_6$ is given by taking the appropriate restrictions.

Proposition 3.2. *Let $\mathbf{D}_6$ act on $\mathscr{C}_{\mathbf{D}_6}$ as in Table 2. Then given $C \in \mathscr{C}_{\mathbf{D}_6}$, the setwise isotropy* $\mathrm{Stab}(C)$ *and the pointwise isotropy* $\mathrm{stab}(C)$ *are listed in Table 3. For each connecting orbit C*

$$\mathrm{Stab}(C)/\mathrm{stab}(C) \cong \mathbf{Z}_2.$$

Table 2. Action of $\mathbf{D}_6$ induced on $\mathscr{C}_{\mathbf{D}_6}$. Only those elements acting nontrivially are shown.

$C \in \mathscr{C}_{\mathbf{D}_6}$	Element of $\mathbf{D}_6$	Action
$(\pi, 0)$	$\rho,\ \rho^4,\ \kappa\rho,\ \kappa\rho^4$	$(0, \pi)$
	$\rho^2,\ \rho^5,\ \kappa,\ \kappa\rho^3$	(π, π)
$(0, \pi)$	$\rho,\ \rho^4,\ \kappa\rho^2,\ \kappa\rho^5$	(π, π)
	$\rho^2,\ \rho^5,\ \kappa\rho,\ \kappa\rho^4$	$(\pi, 0)$
(π, π)	$\rho,\ \rho^4,\ \kappa\rho^3,\ \kappa$	$(\pi, 0)$
	$\rho^2,\ \rho^5,\ \kappa\rho^2,\ \kappa\rho^5$	$(0, \pi)$
$(2\pi/3, 2\pi/3)$	$\rho,\ \rho^3,\ \rho^5,\ \kappa\rho,\ \kappa\rho^3,\ \kappa\rho^5$	$(4\pi/3, 4\pi/3)$
$(4\pi/3, 4\pi/3)$	$\rho,\ \rho^3,\ \rho^5,\ \kappa\rho,\ \kappa\rho^3,\ \kappa\rho^5$	$(2\pi/3, 2\pi/3)$
$C_{(\theta,\theta)}$	$\rho^2,\ \rho^5,\ \kappa\rho^2,\ \kappa\rho^5$	$C_{(2\theta,-\theta)}$
	$\rho,\rho^4,\kappa,\ \kappa\rho^3,\ \rho^5$	$C_{(-\theta,2\theta)}$
$C_{(\theta,0)}$	$\rho,\ \rho^4,\ \kappa\rho,\ \kappa\rho^4$	$C_{(0,\theta)}$
	$\rho^2,\ \rho^5,\ \kappa,\ \kappa\rho^3$	$C_{(\theta,-\theta)}$
$C_{(0,\theta)}$	$\rho,\ \rho^4,\ \kappa\rho^2,\ \kappa\rho^5$	$C_{(\theta,-\theta)}$
	$\rho^2,\ \rho^5,\ \kappa\rho,\ \kappa\rho^4$	$C_{(\theta,0)}$
$C_{(\theta,-\theta)}$	$\rho,\ \rho^4,\ \kappa,\ \kappa\rho^3$	$C_{(\theta,0)}$
	$\rho^2,\ \rho^5,\ \kappa\rho^2,\ \kappa\rho^5$	$C_{(0,\theta)}$
$C_{(2\theta,-\theta)}$	$\rho,\ \rho^4,\ \kappa\rho^2,\ \kappa\rho^5$	$C_{(\theta,\theta)}$
	$\rho^2,\ \rho^5,\ \kappa\rho,\ \kappa\rho^4$	$C_{(-\theta,2\theta)}$
$C_{(-\theta,2\theta)}$	$\rho,\ \rho^4,\ \kappa\rho,\ \kappa\rho^4$	$C_{(2\theta,-\theta)}$
	$\kappa,\ \rho^2,\ \rho^5,\ \kappa\rho^3$	$C_{(\theta,\theta)}$

Proof The computations of Stab(C) are trivial and follow directly from the entries in Table 2. To compute stab(C) we just check to see which elements of Stab(C) act pointwise trivially. The quotient groups follow immediately. □

It is trivial to derive the isotropy data for any subgroup of $\mathbf{D}_6$. We relegate these results to Tables A1–A4 in Appendix A.

Axes of reflection symmetry. A knot relative to a connecting orbit C gives an axis of symmetry for any Δ-equivariant flow on C. When $\Delta = \mathbf{D}_6$, Stab(C)/stab$(C) \cong \mathbf{Z}_2$ for each connecting orbit C, so each C contains exactly two knots, see Theorem 2.3. The only cases where this computation is non-trivial are $C_{(\theta,\theta)}$ and $C_{(\theta,-\theta)}$. Here each connecting orbit contains four equilibria, only two of which are knots. A calculation, the details of which can be found in Parker [17], immediately shows that the knots are those given in Table 4.

When $\Delta = \mathbf{D}_3[\rho^2, \kappa]$ or $\mathbf{D}_3[\rho^2, \kappa\rho]$, Stab$(C)$/stab$(C) \cong \mathbf{1}$ for all $C \in \mathscr{C}_\Delta$, see Tables A1 and A2. Hence there are no knots relative to any connecting orbit.

Table 3. Isotropy data for $C \in \mathscr{C}_{\mathbf{D}_6}$.

$C \in \mathscr{C}_{\mathbf{D}_6}$	$\mathrm{Stab}(C)$	$\mathrm{stab}(C)$
$(0,0)$	$\mathbf{D}_6$	$\mathbf{D}_6$
$(\pi,0)$	$\mathbf{D}_2[\rho^3,\kappa\rho^2]$	$\mathbf{D}_2[\rho^3,\kappa\rho^2]$
$(0,\pi)$	$\mathbf{D}_2[\rho^3,\kappa]$	$\mathbf{D}_2[\rho^3,\kappa]$
(π,π)	$\mathbf{D}_2[\rho^3,\kappa\rho]$	$\mathbf{D}_2[\rho^3,\kappa\rho]$
$(2\pi/3,2\pi/3)$	$\mathbf{D}_3[\rho^2,\kappa]$	$\mathbf{D}_3[\rho^2,\kappa]$
$(4\pi/3,4\pi/3)$	$\mathbf{D}_3[\rho^2,\kappa]$	$\mathbf{D}_3[\rho^2,\kappa]$
$C_{(\theta,\theta)}$	$\mathbf{D}_2[\rho^3,\kappa\rho]$	$\mathbf{Z}_2[\kappa\rho^4]$
$C_{(\theta,0)}$	$\mathbf{D}_2[\rho^3,\kappa\rho^2]$	$\mathbf{Z}_2[\kappa\rho^5]$
$C_{(0,\theta)}$	$\mathbf{D}_2[\rho^3,\kappa]$	$\mathbf{Z}_2[\kappa\rho^3]$
$C_{(\theta,-\theta)}$	$\mathbf{D}_2[\rho^3,\kappa\rho]$	$\mathbf{Z}_2[\kappa\rho]$
$C_{(2\theta,-\theta)}$	$\mathbf{D}_2[\rho^3,\kappa]$	$\mathbf{Z}_2[\kappa]$
$C_{(-\theta,2\theta)}$	$\mathbf{D}_2[\rho^3,\kappa\rho^2]$	$\mathbf{Z}_2[\kappa\rho^2]$

Table 4. Knots relative to the connecting orbits.

$C \in \mathscr{C}_{\mathbf{D}_6}$	Knots
$C_{(\theta,\theta)}$	$(0,0)$, (π,π)
$C_{(\theta,0)}$	$(0,0)$, $(\pi,0)$
$C_{(0,\theta)}$	$(0,0)$, $(0,\pi)$
$C_{(\theta,\theta)}$	$(0,0)$, (π,π)
$C_{(2\theta,-\theta)}$	$(0,0)$, $(\pi,0)$
$C_{(-\theta,2\theta)}$	$(0,0)$, $(0,\pi)$

When $\Delta = \mathbf{D}_2[\rho^3,\kappa]$ Table A3 reveals that $\mathrm{Stab}(C)/\mathrm{stab}(C) \cong \mathbf{Z}_2$, for all connecting orbits C. Therefore, each connecting orbit has two knots, these knots are given by the two equilibria on C. Therefore the knots are $(0,0)$ and $(0,\pi)$ for both Cs.

Finally when $\Delta = \mathbf{D}_2[\rho^3,\kappa\rho]$ the entries of Table A4 show the connecting orbits $C_{(0,\theta)}$ and $C_{(2\theta,-\theta)}$ have two knots; these are the equilibria $(0,0)$ and (π,π).

Remark 1. Not all of the elements of $E_{(\mathbf{D}_6,X_0)}$ are knots. The two points $(2\pi/3,2\pi/3)$ and $(4\pi/3,4\pi/3)$ are not knots relative to any connecting orbit. This answers a question of Lauterbach *et al* [13]. The authors ask whether for general Γ, Σ and Δ all elements of $E_{(\Delta,\Gamma/\Sigma)}$ which are contained in some connecting orbit are knots relative to at least one element of $\mathscr{C}_\Delta$. Here Γ is the symmetry of the unperturbed system, Σ is the symmetry of the steady-state under consideration and Δ is the symmetry of the perturbation. The authors show for $\Gamma = \mathbf{SO}(3)$ this is always true; our analysis shows in general it is false.

3.1.3 ***Projected skeletons.*** Here we compute the projected skeleton corresponding to each skeleton. In each case we determine: 1) a collection of Δ-orbit representatives for the equilibria on the skeleton; 2) use the Δ-action on $\mathscr{C}_\Delta$ to determine orbit representatives for the connecting orbits; 3) given the axes of symmetry on these connecting orbits (i.e. the knots) determine the form of the connecting orbits in the Δ-orbit space. For example, if C has two knots, then C projects into the orbit space as a curve joining those two knots; the reflection symmetry identifies the two arcs either side of the knots. We denote by $\overline{C}$ the projection of $C \in \mathscr{C}_\Delta$ into the Δ-orbit space.

Projected skeleton $\mathbb{X}^p_{\mathbf{D}_6}$. The action of $\mathbf{D}_6$ on $\mathscr{C}_{\mathbf{D}_6}$ shows there are three orbit representatives for points in $E_{(\mathbf{D}_6,X_0)}$, see Table 2. These are given by (say) $(0,0)$, $(\pi,0)$ and $(2\pi/3,2\pi/3)$, and their projection into the $\mathbf{D}_6$-orbit space are: $\overline{(0,0)}$, $\overline{(\pi,0)}$ and $\overline{(2\pi/3,2\pi/3)}$. There are two representatives for the connecting orbits, see Table 2. We take $C_{(\theta,\theta)}$ and $C_{(\theta,0)}$.

The entries of Table 4 show there are two knots relative to $C_{(\theta,\theta)}$ and $C_{(\theta,0)}$, so each have an axis of reflection symmetry. Hence, $C_{(\theta,\theta)}$ and $C_{(\theta,0)}$ both project into the orbit space as a curve joining the two knots; that is, the projection of $C_{(\theta,\theta)}$ is a curve joining $\overline{(0,0)}$ and $\overline{(\pi,0)}$ (the projection of (π,π)), and the projection of $C_{(\theta,0)}$ is a curve joining $\overline{(0,0)}$ and $\overline{(\pi,0)}$. The connecting orbit $C_{(\theta,\theta)}$ contains the elements $(2\pi/3,2\pi/3)$ and $(4\pi/3,4\pi/3)$ which are not knots. Thus the projection of $C_{(\theta,\theta)}$ is a curve joining the representatives $\overline{(0,0)}$ to $\overline{(\pi,0)}$ passing through $\overline{(2\pi/3,2\pi/3)}$.

Figure 2(a) illustrates the projected skeleton. The arrows illustrate an example flow on the projected skeleton.

The analysis of the subsequent cases follows the same lines as above, so we just sketch the details below. Figures 2(b–e) shows the remaining projected skeletons. In each case the arrows show an example flow.

Projected Skeleton $\mathbb{X}^p_{\mathbf{D}_3[\rho^2,\kappa]}$. Table 2 shows that $(0,0)$, $(2\pi/3,2\pi/3)$ and $(4\pi/3,4\pi/3)$ are orbit representatives for the equilibria and there is one orbit representative for the connecting orbits, we take $C_{(\theta,\theta)}$. There are no knots on the connecting orbit, so no axes of reflection symmetry. Therefore, the projection of the connecting orbit into the orbit space is a topological circle joining $\overline{(0,0)}$, $\overline{(2\pi/3,2\pi/3)}$ and $\overline{(4\pi/3,4\pi/3)}$.

Projected Skeleton $\mathbb{X}^p_{\mathbf{D}_3[\rho^2,\kappa\rho]}$. Table 2 shows that $(0,0)$ and $(2\pi/3, 2\pi/3)$ are orbit representatives for the equilibria and there is one orbit representative for the connecting orbits, $C_{(\theta,0)}$ say. There are no knots on the connecting orbit so no axes of reflection symmetry. Therefore, the projection of $C_{(\theta,0)}$ into the orbit space is a circle with the equilibria $\overline{(0,0)}$ lying on it. The equilibria $\overline{(2\pi/3, 2\pi/3)}$ is isolated on the projected skeleton; that is, it does not lie on any connecting orbit.

Projected skeleton $\mathbb{X}^p_{\mathbf{D}_2[\rho^3,\kappa]}$. The orbit representatives for the equilibria are : $(0,0)$, $(\pi,0)$, and (π,π) and there are two orbit representatives for the connecting orbits given by $C_{(0,\theta)}$ and $C_{(-\theta,2\theta)}$. There are two knots on these connecting orbits given by $(0,0)$ and $(\pi,0)$. The projections of the connecting orbits $C_{(0,\theta)}$ and $C_{(-\theta,2\theta)}$ are distinct curves connecting $\overline{(0,0)}$ to $\overline{(\pi,0)}$.

Projected skeleton $\mathbb{X}^p_{\mathbf{D}_2[\rho^3,\kappa\rho]}$. This case is identical to the previous one. The orbit space representatives for the equilibria are given by $\overline{(0,0)}$, $\overline{(\pi,0)}$ and $\overline{(\pi,\pi)}$. The representatives for the connecting orbits are: $C_{(\theta,\theta)}$ and $C_{(\theta,-\theta)}$. The connecting orbits project to distinct curves joining $\overline{(0,0)}$ and $\overline{(\pi,\pi)}$.

Remark 2 1) Since the equilibrium $(2\pi/3, 2\pi/3)$ is not a knot, there exist flows on the projected skeleton $\mathbb{X}^p_{\mathbf{D}_6}$ such that $\overline{(2\pi/3, 2\pi/3)}$ is not hyperbolic. However, it is possible to exhibit flows for which all the equilibria are hyperbolic as seen in Figure 2(a).
2) It is not possible to find any admissible perturbation on the projected skeleton $\mathbb{X}^p_{\mathbf{D}_3[\rho^2,\kappa]}$ for which all equilibria are hyperbolic, such a non-hyperbolic flow is illustrated in Figure 2(b).
3) Any admissible perturbation on $\mathbb{X}^p_{\mathbf{D}_3[\rho^2,\kappa\rho]}$, leads to a homoclinic cycle on the projected skeleton, see Figure 2(c).

We rewrite point (3) of Remark *2* in the following proposition.

PROPOSITION 3.3 *There exists a* $\mathbf{D}_3[\rho^2,\kappa\rho]$*-equivariant perturbation of a* $\mathbf{D}_6 \dot{+} \mathbf{T}^2$*-equivariant bifurcation problem that gives homoclinic cycles for the perturbed flow.*

Proof Theorem 2.4 guarantees the existence of these flows. □

Subsequently we give linear order conditions on the coefficients of the Taylor expansion of a $\mathbf{D}_3[\rho^2,\kappa]$-equivariant perturbation that guarantee these homoclinic cycles exist.

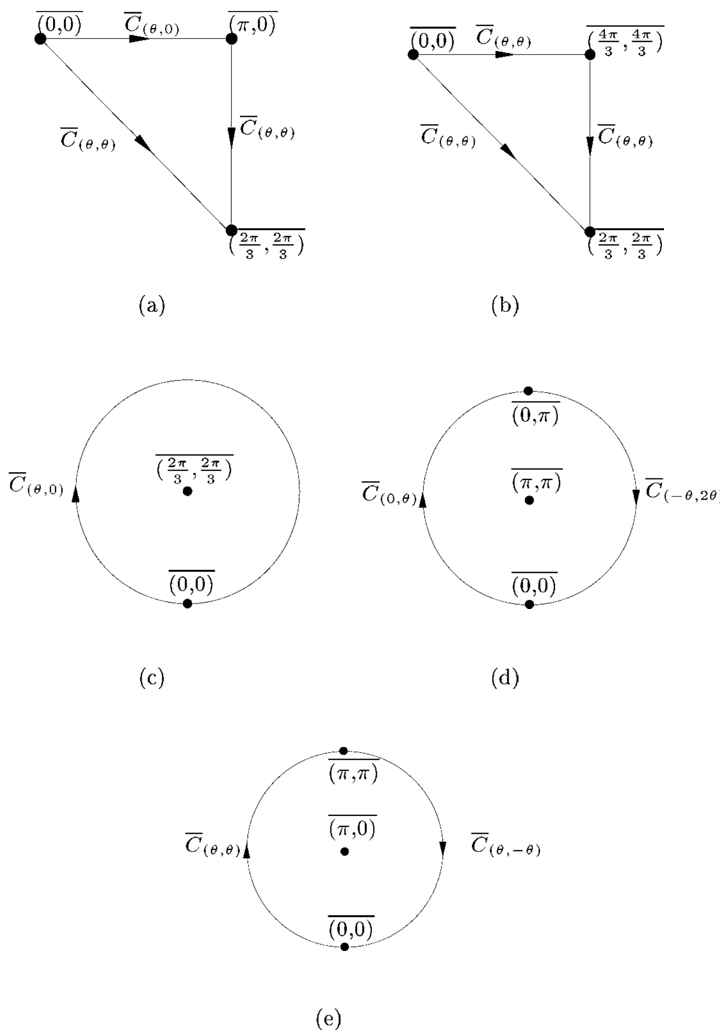


Figure 2. The projected skeletons. The arrows illustrate possible flows.

3.2 *Heteroclinic Cycles on the Projected Skeletons*

In this section we determine conditions on the low order terms in the Taylor expansion of certain perturbations that give admissible flows. Since the projected skeletons $\mathbb{X}^p_{\mathbf{D}_6}$ and $\mathbb{X}^p_{\mathbf{D}_3[\rho^2,\kappa]}$ both can give rise to non-hyperbolic equilibria, we do not consider these cases.

3.2.1 ***Flows on*** $\mathbb{X}^p_{\mathbf{D}_3[\rho^2,\kappa\rho]}$ **.** The equilibria $(0,0)$ is contained on the connection $C_{(\theta,0)}$. We parametrise $C_{(\theta,0)}$ by the function

$$\omega(t) = (e^{\mathrm{i}t}, 1, e^{-\mathrm{i}t})$$

for $t \in [0, 2\pi]$. The tangent vector to $\omega(t)$ is

$$\mathcal{T}(t) = (-\sin t, \cos t, 0, 0, -\sin t, -\cos t).$$

The following proposition reduces the types of admissible perturbations.

PROPOSITION 3.4 *Let* $g : \mathbb{C}^3 \to \mathbb{C}^3$ *be a* $\mathbf{D}_3[\rho^2, \kappa\rho]$*-equivariant vector field. Suppose all the coefficients of* g *are real. Then* g *is not an admissible perturbation.*

Proof Let $g = (g_1, g_2, g_3)$ be $\mathbf{D}_3[\rho^2, \kappa\rho]$-equivariant. It is only necessary to consider the first component of g. The first component has the form

$$g_1(z) = \sum a_{\alpha\beta} z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} \overline{z_1}^{\beta_1} \overline{z_2}^{\beta_2} \overline{z_3}^{\beta_3},$$

where α_i, β_i are integers and $a_{\alpha\beta} \in \mathbb{R}$.

Evaluating g along the connection $C_{(\theta,0)}$ we find that

$$g_1(\omega(t)) = g_1(e^{\mathrm{i}t}, 1, e^{-\mathrm{i}t}) = \sum a_{\alpha\beta} \left(\cos(\alpha_1 - \alpha_3)t\right) + \mathrm{i} \sum a_{\alpha\beta} \sin\left((\beta_1 - \beta_3)t\right),$$

$$g_3(\omega(t)) = g_1(e^{-\mathrm{i}t}, e^{\mathrm{i}t}, 1) = \sum a_{\alpha\beta} \left(\cos(-\alpha_1 + \alpha_2)t\right) + \mathrm{i} \sum a_{\alpha\beta} \sin\left((-\beta_1 + \beta_2)t\right),$$

The second equality follows from the equivariance condition $g_3(z) = g_1(z_3, z_1, z_2)$, the splitting into real and imaginary parts follows since $a_{\alpha\beta} \in \mathbb{R}$. We have not evaluated g_2 since this term makes no contribution to the flow formula.

The flow formula is

$$\begin{aligned}\mathcal{F}(t) &= \langle g(\omega(t)), \mathcal{T}(t) \rangle \\ &= -\sin t \left(\sum a_{\alpha\beta} \left(\cos(\alpha_1 - \alpha_3)t + \cos(-\alpha_1 + \alpha_2)t\right) \right) \\ &\quad - \cos t \left(\sum a_{\alpha\beta} \left(\sin(\beta_1 - \beta_3)t\right) + \sin\left((-\beta_1 + \beta_2)t\right) \right).\end{aligned}$$

Thus $\mathcal{F}(\pi) = 0$ and the perturbation is not admissible. □

Remark 3. This proposition holds for any $\mathbf{D}_3[\rho^2, \kappa\rho]$-equivariant map, and not just some low order truncation. Thus a sufficient condition for homoclinic cycles on the projected skeleton is that the perturbation term must have some non-real coefficients.

The remainder of this section classifies those linear perturbations that are admissible—and by implication give homoclinic cycles. The Poincaré series (see Appendix B) shows there are 27 linear and quadratic equivariants, hence our restriction to linear order.

The computations in Appendix B show that to linear order the general form of a $\mathbf{D}_3[\rho^2, \kappa\rho]$-equivariant mapping is:

$$g_1(z) = p_1 u_1 + p_2 z_1 + p_3 \overline{z_1} + p_4 z_2 + \overline{p_4} z_3 + p_5 \overline{z_2} + \overline{p_5 z_3}, \tag{11}$$

where $u_1 = z_1 + z_2 + z_3 + \overline{z_1} + \overline{z_2} + \overline{z_3}$. The coefficients p_4 and p_5 are complex and all other coefficients are real. The other components of the vector field are given by equivariance. By considering each equivariant in (11) we determine which equivariants give admissible flows on the skeleton. The linearity of $\mathcal{F}(t)$ gives the flow formulae for any perturbation involving these terms. We do not need to consider the equivariant (z_1, z_2, z_3) since it is Γ-equivariant, so induces trivial flow. Furthermore, the equivariance condition $\overline{g_1(z_1, z_2, z_3)} = g_1(\overline{z_1}, \overline{z_3}, \overline{z_2})$ implies the coefficients of the perturbations (u_1, u_1, u_1) and $(\overline{z_1}, \overline{z_2}, \overline{z_3})$ are real, hence by Proposition 3.4 they do not induce admissible flows. We consider the two remaining equivariants.

PROPOSITION 3.5. *Define* $g : \mathbb{C}^3 \to \mathbb{C}^3$ *by* $g(z) = (cz_2 + \bar{c}z_3, cz_1 + \bar{c}z_3, cz_2 + \bar{c}z_1)$, *where* $c = a + ib \in \mathbb{C}$. *Then the flow formula is*

$$\mathcal{F}(t) = 4x^2 \sin(3t/2) \left(-a \cos\left(\frac{t}{2}\right) + b \sin\left(\frac{t}{2}\right) \right).$$

This flow has zeros at $t = 2\pi/3$ *and* $t = 4\pi/3$ *and so is not admissible.*

Of course the flow can have other zeros, but we are only concerned with finding zeros that forbid admissible flows.

Proof Define $g(z) = (cz_2 + \bar{c}z_3, cz_3 + \bar{c}z_1, cz_2 + \bar{c}z_1)$. Then (not evaluating the g_2 component, which is not required)

$$\begin{aligned} g(\omega(t)) &= ((a + ib) + (a - ib)(\cos t - i \sin t), g_2, \\ &\quad (a + ib)(\cos t + i \sin t) + (a - ib)) \\ &= (a + a\cos t - b\sin t, b - b\cos t - a\sin t, g_2, \\ &\quad a\cos t - b\sin t + a, b\cos t + a\sin t - b)\,. \end{aligned}$$

Hence the flow formula is

$$\begin{aligned}\mathcal{F}(t) &= (-\sin t(a + a\cos t - b\sin t) + \cos t(b - b\cos t - a\sin t) \\ &\quad - \sin t(a\cos t - b\sin t + a) - \cos t(b\cos t + a\sin t - b))x^2 \\ &= (-2a\sin t + 2b\cos t - 4a\sin t\cos t + 2b\sin^2 t - 2b\cos^2 t)x^2 \\ &= 4x^2\sin(3t/2)\left(-a\cos\left(\frac{t}{2}\right) + b\sin\left(\frac{t}{2}\right)\right),\end{aligned}$$

with the final equality following from some tedious algebra. The location of the claimed zeros is easily seen from the $\sin(3t/2)$ term. □

There is one linear perturbation remaining as a candidate for an admissible perturbation.

Proposition 3.6. *Define* $g : \mathbb{C}^3 \to \mathbb{C}^3$ *by* $g(z) = (d\overline{z_3} + \overline{d}\overline{z_2}, d\overline{z_1} + \overline{d}\overline{z_3}, d\overline{z_2} + \overline{d}\overline{z_1})$*, where* $d = a + \mathrm{i}b \in \mathbb{C}$*. Then the flow formula is*

$$\mathcal{F}(t) = -2x^2\left(a\sin t + b(\cos t - 1)\right).$$

If $a = 0$ *and* $b \neq 0$ *then* g *is admissible. If* $a \neq 0$ *then there is a zero of the flow formula in the interval* $(0, 2\pi)$*, so* g *is not admissible.*

Proof Define $g(z) = (d\overline{z_3} + \overline{d}\overline{z_2}, d\overline{z_1} + \overline{d}\overline{z_3}, d\overline{z_2} + \overline{d}\overline{z_1})$. Then (not evaluating the g_2 component, which is not required) we find

$$\begin{aligned}g(\omega(t)) &= ((a - \mathrm{i}b) + (a + \mathrm{i}b)(\cos t + \mathrm{i}\sin t), g_2, \\ &\quad (a - \mathrm{i}b)(\cos t - \mathrm{i}\sin t) + (a + \mathrm{i}b)) \\ &= (a + a\cos t - b\sin t, -b + b\cos t + a\sin t, g_2, \\ &\quad a\cos t - b\sin t + a, -b\cos t - a\sin t + b).\end{aligned}$$

Hence the flow formula is

$$\begin{aligned}\mathcal{F}(t) &= (-\sin t(a + a\cos t - b\sin t) + \cos t(-b + b\cos t + a\sin t) \\ &\quad - \sin t(a\cos t - b\sin t + a) - \cos t(-b\cos t - a\sin t + b))x^2 \\ &= -2x^2\left(a\sin t + b(\cos t - 1)\right).\end{aligned}$$

Obviously if $a = 0$ and $b \neq 0$ then g is admissible. If $a \neq 0$ and $b = 0$ then g is obviously not admissible. We now claim if $a \neq 0$ and $b \neq 0$ then $\mathcal{F}(t)$ has a zero in the interval $(0, 2\pi)$. A computation shows if $a \neq \pm b$ then $\mathcal{F}(t)$ has zeros located at

$$\pm \tan^{-1}\left(\frac{2ab}{b^2-a^2}\right).$$

If $a, b \neq 0$ then one of these zeros is located in $(0, 2\pi)$. This shows g is not admissible. Finally if $a = \pm b$, then it is trivial to see that g is not admissible. □

THEOREM 3.7 *There exists an open set of* $\mathbf{D}_3[\rho^2, \kappa\rho]$*-equivariant perturbations of a* $\mathbf{D}_6 \dot{+} \mathbf{T}^2$*-equivariant bifurcation problem that give a homoclinic cycle for the perturbed flow.*

Proof By Proposition 3.6 there exists an open set of $\mathbf{D}_3[\rho^2, \kappa\rho]$-equivariant perturbations that are admissible. These perturbations give a homoclinic cycle on the skeleton. By Theorem 2.7 this homoclinic cycle persist on the perturbed skeleton for sufficiently small perturbations. □

3.2.2 $\mathbf{D}_2[\rho^3, \kappa]$***- and*** $\mathbf{D}_2[\rho^3, \kappa\rho]$***-equivariant perturbations.*** A general classification of the flow formulae on the two remaining skeletons is too complex to undertake; there are too many different possibilities even at quadratic order, the main problem is that one component of the vector field is always unrelated to the other two. Instead we provide examples illustrating heteroclinic cycles on the projected skeletons $\mathbb{X}^p_{\mathbf{D}_2[\rho^3,\kappa]}$ and $\mathbb{X}^p_{\mathbf{D}_2[\rho^3,\kappa\rho]}$.

The $\mathbf{D}_2[\rho^3, \kappa]$**-equivariant case.** Define $g : \mathbb{C}^3 \to \mathbb{C}^3$ by

$$g(z) = (g_1, g_2, g_3) = (az_1(z_2 + z_3), b(z_2\overline{z_2} + z_3\overline{z_3}), b(z_2\overline{z_2} + z_3\overline{z_3})).$$

Then g is $\mathbf{D}_2[\rho^3, \kappa]$-equivariant.

The non-isolated equilibria on the projected skeleton are $\overline{(0,0)}$ and $\overline{(0,\pi)}$. These equilibria lie at the end points of the connections $\overline{C}_{(0,\theta)}$ and $\overline{C}_{(2\theta,-\theta)}$. Thus we need only consider the following connections on the skeleton:

$$C_{(0,\theta)} \text{ and } C_{(2\theta,-\theta)}.$$

Specifically we need only consider the flow on the following subsets of these connections:

$$\vec{C}_{(0,\theta)} := \{(0,\theta) : \theta \in (0,\pi)\}$$
$$\vec{C}_{(2\theta,-\theta)} := \{(2\theta,-\theta) : \theta \in (0,\pi)\},$$

since the flow on these subsets characterizes the flow on the entire skeleton. A quick calculation, like that performed above, shows that

$$\mathcal{F}_{\overrightarrow{C}_{(0,\theta)}}(t) = -4x^3 b \sin t$$

and

$$\mathcal{F}_{\overrightarrow{C}_{(2\theta,-\theta)}}(t) = -4x^3(a+b)\sin t.$$

The flow along $\overrightarrow{C}_{(2\theta,-\theta)}$ is admissible if $b \neq 0$; the flow along $\overrightarrow{C}_{(0,\theta)}$ is admissible provided $(a+b) \neq 0$. For heteroclinic cycles we require $\mathrm{sgn}(b) = -\mathrm{sgn}(a+b)$.

By taking suitable linear combinations with other $\mathbf{D}_2[\rho^3, \kappa]$-equivariants we can exhibit different perturbations that produce heteroclinic cycles, provided the perturbation g dominates.

The $\mathbf{D}_2[\rho^3, \kappa\rho]$-equivariant case. The mapping $g : \mathbb{C}^3 \to \mathbb{C}^3$ defined by

$$g(z) = (a(z_1\overline{z_1} + z_2\overline{z_2}), a(z_1\overline{z_1} + z_2\overline{z_2}), b(z_3(z_1 + z_2))),$$

where a and b are real, is $\mathbf{D}_2[\rho^3, \kappa\rho]$-equivariant. This case is similar to the previous one, so we will be brief.

The non-isolated equilibria on the projected skeleton are $\overline{(0,0)}$ and $\overline{(\pi,\pi)}$. These equilibria lie at the end points of the connections $\overline{C}_{(\theta,\theta)}$ and $\overline{C}_{(\theta,-\theta)}$. The group action implies we need only consider the flow on the following subsets of these connections:

$$\overrightarrow{C}_{(\theta,\theta)} := \{(\theta,\theta) : \theta \in (0,\pi)\}$$
$$\overrightarrow{C}_{(\theta,-\theta)} := \{(\theta,-\theta) : \theta \in (0,\pi)\}.$$

Along the connection $\overrightarrow{C}_{(\theta,\theta)}$ the flow formula is

$$\mathcal{F}_{\overrightarrow{C}_{(\theta,\theta)}}(t) = -4x^3(a+b)\sin t.$$

Similarly

$$\mathcal{F}_{\overrightarrow{C}_{(\theta,-\theta)}}(t) = -4x^3 a \sin t.$$

For heteroclinic cycles we require a and $a+b$ to be non-zero and $\mathrm{sgn}(a) =$

$-\mathrm{sgn}(a+b)$. Such cycles exist for any combination of g with other $\mathbf{D}_2[\rho^3, \kappa\rho]$-equivariant perturbations provided the g term dominates.

3.3 *Forced Symmetry-breaking of Superhexagons*

The group orbit X_0 given by Proposition 2.9 is a 2-torus. Moreover, X_0 is diffeomorphic to the homogeneous space $\Gamma/\mathbf{D}_6$. The space $\Gamma/\mathbf{D}_6$ is independent of the representation of Γ, we only require that $\mathbf{D}_6$ is an isotropy subgroup in each case. Thus the natural action of $\mathbf{D}_6$ on $\Gamma/\mathbf{D}_6$ is the same in the six- and twelve-dimensional representations of Γ. This implies the skeleton, knots, setwise and pointwise isotropy subgroups are identical in both representations of Γ. This discussion leads to the following result:

THEOREM 3.8. *Let* $\Gamma = \mathbf{D}_6 \dot{+} \mathbf{T}^2$, *and* Δ *be one of the groups* $\mathbf{D}_6$, $\mathbf{D}_3[\rho^2, \kappa]$, $\mathbf{D}_3[\rho^2, \kappa\rho]$, $\mathbf{D}_2[\rho^3, \kappa]$, $\mathbf{D}_2[\rho^3, \kappa\rho]$. *Let* Γ *act on* $\mathbb{C}^6$ *as in (7). Let* f *be a* Γ-*equivariant bifurcation problem. Let* g *be a* Δ-*equivariant vector field which satisfies* $g(0) = 0$. *Then there exists a branch of steady-state solutions to* $f(z, \lambda) = 0$ *bifurcating from the origin with* $\mathbf{D}_6$-*isotropy. Let* $X_0 \cong \mathbf{T}^2$ *be the group orbit of steady states. Consider the perturbed vector field* $P(z, \lambda, \varepsilon) = f(z, \lambda) + \varepsilon g(z)$, *where* ε *is real and small. Then, for sufficiently small* ε, X_0 *persists to give a new* P-*invariant manifold* X_ε, *which is* Δ-*equivariantly diffeomorphic to* X_0. *The behaviour of the vector field on* X_0 *is characterised by the projected skeletons in Figure 2. In particular, homoclinic cycles can occur when the perturbation term has* $\mathbf{D}_3[\rho^2, \kappa\rho]$ *symmetry.*

Proof This follows from the discussion above and the results for the six-dimensional representation. □

It is worth emphasizing that we have proved the existence of a homoclinic cycle when the perturbation has $\mathbf{D}_3[\rho^2, \kappa\rho]$ symmetry. Further investigation is required to determine what type of $\mathbf{D}_3[\rho^2, \kappa\rho]$-equivariant perturbations give rise to this homoclinic cycle, and whether it is stable. Preliminary investigations failed to reveal this cycle in low-order perturbations.

This classification represents the most general statement we can sensibly give; the invariant theory is too complex to allow anything more than the study of specific examples.

4 Discussion and Conclusion

This paper presents a partial classification of the forced symmetry-breaking of hexagon and superhexagon planforms. All projected skeletons were constructed, classifying all equilibria and heteroclinic connections that can arise from the residual symmetry. For both the six- and twelve-dimensional representations homoclinic cycles on the projected skeleton $\mathbb{X}^{p}_{\mathbf{D}_3[\rho^2,\kappa\rho]}$ are guaranteed to exist for certain perturbations. Moreover, explicit construct of the linear order $\mathbf{D}_3[\rho^2, \kappa\rho]$-equivariant vector field in the six-dimensional representation allowed us to prove that homoclinic cycles exist for an open set of $\mathbf{D}_3[\rho^2, \kappa\rho]$-equivariant vector fields, although the stability of this cycle was not considered. Examples were provided showing that heteroclinic cycles exist for $\mathbf{D}_2[\rho^3, \kappa]$- and $\mathbf{D}_2[\rho^3, \kappa\rho]$-equivariant perturbations, although the stability of these cycles was not studied. All results concerning the persistence of equilibria and the existence of heteroclinic connections are essentially identical for the six- and twelve-dimensional representations. The invariants and equivariants in the twelve-dimensional representation are complex, so it is not possible to classify flows on the skeletons. Nevertheless, we have qualitatively enumerated all the heteroclinic behaviour that can occur via forced symmetry-breaking (to the groups discussed).

The results of this paper represent a generalisation of those of Hou and Golubitsky [15] and Parker *et al.* [16]. These papers study the perturbation of square lattice planforms. Hou and Golubitsky [15] prove there exists an open set of perturbations that give heteroclinic cycles for the perturbed flow. Parker *et al.* [16] generalised these results to other perturbation, classifying the admissible perturbation up to quadratic order (within the class of perturbations under consideration). The eight-dimensional bifurcation problem on the square lattice is studied in the same way as the twelve-dimensional representation on the hexagonal lattice, and the results are again limited: only the skeletons and the symmetry of their components is determined. In general it is difficult to give general results for the high-dimensional bifurcation problems due to the complex invariant theory. The hexagonal problem demonstrated that not all equilibria that lie on a connecting orbit are necessarily knots relative to that orbit, see Remark *1*. The square lattice problem does not possess this difficulty.

Hexagonal patterns arise in many natural phenomenon including Faraday waves [10, 24], Rayleigh–Bénard convection [1, 22, 23] and reaction-diffusion systems [1]. Indeed, hexagonal patterns may arise in any system where periodic or Neumann boundary conditions are assumed. Our results apply directly to these systems where the boundary conditions are slightly perturbed. At present our results only represent a qualitative classification of the different types of behaviour. Nevertheless, the results of this paper can be used to examine the effects of distant sidewalls on pattern formation in these systems. Indeed, any system exhibiting steady hexagonal patterns falls under the purview of our results.

Further analysis is required to determine the genericity of our results and the stability of the homoclinic cycles. In particular, quantitative data can only be derived by a full Liapunov–Schmidt reduction of the modelling equations of a chosen system. The details of this are beyond the scope of this paper. A study of three-dimensional planforms has been considered [17].

Appendix A: Isotropy Data

Tables A1–A4 contain the isotropy data for the elements of $\mathscr{C}_{\mathbf{D}_3[\rho^2,\kappa]}$, $\mathscr{C}_{\mathbf{D}_3[\rho^2,\kappa\rho]}$, $\mathscr{C}_{\mathbf{D}_2[\rho^3,\kappa]}$ and $\mathscr{C}_{\mathbf{D}_2[\rho^3,\kappa\rho]}$.

Appendix B: $\mathbf{D}_3[\rho^2,\kappa\rho]$-Invariants and -Equivariants

The determination of the flow formula for a general $\mathbf{D}_3[\rho^2,\kappa\rho]$-equivariant map requires the invariant functions and equivariant maps. Here we compute these up to linear order. The theorems of Schwarz and Poénaru [2] imply it is sufficient to consider only polynomial functions and maps. All computations were performed using Maple Version 7™. For further details see Parker [17].

Table A1. Isotropy data for $C \in \mathscr{C}_{\mathbf{D}_3[\rho^2,\kappa]}$.

$C \in \mathscr{C}_{\mathbf{D}_3[\rho^2,\kappa]}$	Stab(C)	stab(C)
$(0,0)$	$\mathbf{D}_3[\rho^2,\kappa]$	$\mathbf{D}_3[\rho^2,\kappa]$
$(2\pi/3,2\pi/3)$	$\mathbf{D}_3[\rho^2,\kappa]$	$\mathbf{D}_3[\rho^2,\kappa]$
$(4\pi/3,4\pi/3)$	$\mathbf{D}_3[\rho^2,\kappa]$	$\mathbf{D}_3[\rho^2,\kappa]$
$C_{(\theta,\theta)}$	$\mathbf{Z}_2[\kappa\rho^4]$	$\mathbf{Z}_2[\kappa\rho^4]$
$C_{(2\theta,-\theta)}$	$\mathbf{Z}_2[\kappa]$	$\mathbf{Z}_2[\kappa]$
$C_{(-\theta,2\theta)}$	$\mathbf{Z}_2[\kappa\rho^2]$	$\mathbf{Z}_2[\kappa\rho^2]$

Table A2. Isotropy data for $C \in \mathscr{C}_{\mathbf{D}_3[\rho^2,\kappa\rho]}$.

$C \in \mathscr{C}_{\mathbf{D}_3[\rho^2,\kappa\rho]}$	Stab(C)	stab(C)
$(0,0)$	$\mathbf{D}_3[\rho^2,\kappa]$	$\mathbf{D}_3[\rho^2,\kappa]$
$(2\pi/3,2\pi/3)$	$\mathbf{Z}_3[\rho^2]$	$\mathbf{Z}_3[\rho^2]$
$(4\pi/3,4\pi/3)$	$\mathbf{Z}_3[\rho^2]$	$\mathbf{Z}_3[\rho^2]$
$C_{(\theta,0)}$	$\mathbf{Z}_2[\kappa\rho^5]$	$\mathbf{Z}_2[\kappa\rho^5]$
$C_{(0,\theta)}$	$\mathbf{Z}_2[\kappa\rho^3]$	$\mathbf{Z}_2[\kappa\rho^3]$
$C_{(\theta,-\theta)}$	$\mathbf{Z}_2[\kappa\rho]$	$\mathbf{Z}_2[\kappa\rho]$

Table A3. Isotropy data for $C \in \mathscr{C}_{\mathbf{D}_2[\rho^3,\kappa]}$.

$C \in \mathscr{C}_{\mathbf{D}_2[\rho^3,\kappa]}$	Stab(C)	stab(C)
$(0,0)$	$\mathbf{D}_2[\rho^3,\kappa]$	$\mathbf{D}_2[\rho^3,\kappa]$
$(\pi,0)$	$\mathbf{Z}_2[\rho^3]$	$\mathbf{Z}_2[\rho^3]$
$(0,\pi)$	$\mathbf{D}_2[\rho^3,\kappa]$	$\mathbf{D}_2[\rho^3,\kappa]$
(π,π)	$\mathbf{Z}_2[\rho^3]$	$\mathbf{Z}_2[\rho^3]$
$C_{(0,\theta)}$	$\mathbf{D}_2[\rho^3,\kappa]$	$\mathbf{Z}_2[\kappa\rho^3]$
$C_{(2\theta,-\theta)}$	$\mathbf{D}_2[\rho^3,\kappa]$	$\mathbf{Z}_2[\kappa]$

Table A4. Isotropy data for $C \in \mathscr{C}_{\mathbf{D}_2[\rho^3,\kappa\rho]}$.

$C \in \mathscr{C}_{\mathbf{D}_2[\rho^3,\kappa\rho]}$	Stab(C)	stab(C)
$(0,0)$	$\mathbf{D}_2[\rho^3,\kappa\rho]$	$\mathbf{D}_2[\rho^3,\kappa\rho]$
$(\pi,0)$	$\mathbf{Z}_2[\rho^3]$	$\mathbf{Z}_2[\rho^3]$
$(0,\pi)$	$\mathbf{Z}_2[\rho^3]$	$\mathbf{Z}_2[\rho^3]$
(π,π)	$\mathbf{D}_2[\rho^3,\kappa\rho]$	$\mathbf{D}_2[\rho^3,\kappa\rho]$
$C_{(\theta,\theta)}$	$\mathbf{D}_2[\rho^3,\kappa\rho]$	$\mathbf{Z}_2[\kappa\rho^4]$
$C_{(\theta,-\theta)}$	$\mathbf{D}_2[\rho^3,\kappa\rho]$	$\mathbf{Z}_2[\kappa\rho]$

We begin with the invariants. The Poincaré series for the invariants is

$$\begin{aligned}\Upsilon(t) &= \frac{t^6+t^5+5t^4+3t^3+t^2+1}{(1-t^3)^2(1-t^2)^3(1-t)} \\ &= 1+t+5t^2+10t^3+24t^4+42t^5+\dots,\end{aligned}$$

where ... denotes higher order terms. Using the Poincaré series there is one linear order invariant:

$$u_1 = z_1+z_2+z_3+\overline{z_1}+\overline{z_2}+\overline{z_3},$$

The quadratic and higher order invariants are not required for our work and are ignored, they can be found in Parker [17].

The Poincaré series for the equivariants is

$$\begin{aligned}\Xi(t) &= \frac{1}{(1-t)^6} \\ &= 1+6t+21t^2+56t^3+126t^4+252t^5+\dots,\end{aligned}$$

where ... denotes higher order terms. The polynomial maps

$$\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}z_1\\z_2\\z_3\end{bmatrix}, \begin{bmatrix}wz_2+\overline{w}z_3\\wz_3+\overline{w}z_1\\wz_1+\overline{w}z_2\end{bmatrix}, \begin{bmatrix}\overline{z_1}\\\overline{z_2}\\\overline{z_3}\end{bmatrix}, \begin{bmatrix}w\overline{z_2}+\overline{wz_3}\\w\overline{z_3}+\overline{wz_1}\\w\overline{z_1}+\overline{wz_2}\end{bmatrix},$$

where $w \in \mathbb{C}$ are equivariant under the $\mathbf{D}_3[\rho^2, \kappa\rho]$-action. Furthermore, they generate the module of polynomial equivariants over the primary invariants, up to linear order.

It would appear from the Poincaré series for the equivariants that there is a linear order equivariant missing; this is not so, but results from the choice of coordinates used to compute the Poincaré series.

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